Math 5101, NEU, Fall 2018
Instructor: Daniel Glasscock Name:

Homework #1A

Due September 20, 2018
Peer evaluation due October 2, 2018

Instructions: If you choose to submit the problems on this side, staple this sheet to your work so that this is the first page. Your original work will be given to one of your peers for evaluation, so you are strongly encouraged to make a copy of it before submission.

Unless otherwise specified, (X, d) is a metric space.

1A. Let $Y \subseteq X$ be a metric subspace of (X, d). Prove or disprove: A set $B \subseteq Y$ is closed in (Y, d) if and only if there exists a closed subset A of (X, d) for which $A \cap Y = B$.

2A. Definition: Two metrics d_1 and d_2 on a set X are *strongly equivalent* if there exist $c_1, c_2 > 0$ so that for all $x, y \in X$,

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y).$$

Let d_1 and d_2 be strongly equivalent metrics on the set X.

i. Let $A \subseteq X$. Prove that A is open with respect to d_1 if and only if it is open with respect to d_2. (We say that the metrics *have the same open sets.*)

ii. Let $(x_n)_{n \in \mathbb{N}} \subseteq X$ and $x \in X$. Prove that $\lim_{n \to \infty} x_n = x$ with respect to d_1 if and only if $\lim_{n \to \infty} x_n = x$ with respect to d_2.

iii. Give an example of two metrics on $\mathbb{R}$ that have the same open sets but are not strongly equivalent.

3A. Definition: A subset $A \subseteq X$ is *disconnected* if there exist non-empty subsets $A_1, A_2 \subseteq X$ such that $A = A_1 \cup A_2$ and $\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \emptyset$. The metric space (X, d) is *disconnected* if X is a disconnected subset of (X, d).

i. Let $A, U \subseteq X$. Prove that if U is open and $A \cap U = \emptyset$, then $\overline{A} \cap U = \emptyset$.

ii. Prove that $A \subseteq X$ is disconnected if and only if there exist open sets $U, V \subseteq X$ such that $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$, $A \subseteq U \cup V$, and $A \cap U \cap V = \emptyset$.

4A. Let $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$ be Cauchy sequences. Prove that $\big(d(x_n, y_n)\big)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Using that $\mathbb{R}$ is complete, conclude that $\big(d(x_n, y_n)\big)_{n \in \mathbb{N}}$ converges, even if neither $(x_n)_{n \in \mathbb{N}}$ nor $(y_n)_{n \in \mathbb{N}}$ converge.

5A. Prove or disprove: A metric subspace of a separable metric space is separable.

Math 5101, NEU, Fall 2018
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Homework #1B

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Unless otherwise specified, (X, d) is a metric space.

1B. Let $Y \subseteq X$ be a metric subspace of (X, d). Prove or disprove: If A is a closed subset of Y and Y is a closed subset of X, then A is a closed subset of X.

2B. Definition: Two metrics d_1 and d_2 on a set X are *strongly equivalent* if there exist $c_1, c_2 > 0$ so that for all $x, y \in X$,

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y).$$

Let $X = \{0, 1\}^{\mathbb{N}}$, the space of all sequences $x = (x_n)_{n \in \mathbb{N}} \subseteq \{0, 1\}$, and let

$$d_1(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-\min\{n \in \mathbb{N} \mid x_n \neq y_n\}} & \text{if } x \neq y \end{cases},$$
$$d_2(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

Show that d_1 and d_2 are metrics on X and that they are strongly equivalent.

3B. Definition: A subset $A \subseteq X$ is *disconnected* if there exist non-empty subsets $A_1, A_2 \subseteq X$ such that $A = A_1 \cup A_2$ and $\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \emptyset$. The metric space (X, d) is *disconnected* if X is a disconnected subset of (X, d).

i. Let $B \subseteq A$. Denote by $\overline{B}^X$ and $\overline{B}^A$ the closure of B as a subset of the metric spaces (X, d) and (A, d), respectively. Prove that $\overline{B}^A = \overline{B}^X \cap A$.

ii. Prove that A is disconnected as a subset of (X, d) if and only if (A, d) is a disconnected metric space. (That is, the property of being disconnected is *intrinsic*, and hence so is the property of being *connected*.)

4B. Let $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$. Prove that if $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, then $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$.

5B. Let $(x_n)_{n \in \mathbb{N}} \subseteq X$ and $x \in X$. Prove or disprove: If $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and a subsequence of $(x_n)_{n \in \mathbb{N}}$ converges to x, then $(x_n)_{n \in \mathbb{N}}$ converges to x.

Math 5101, NEU, Fall 2018
Instructor: Daniel Glasscock Name:

Homework #2A

Due October 9, 2018
Peer evaluation due October 18, 2018

Instructions: If you choose to submit the problems on this side, staple this sheet to your work so that this is the first page. Your original work will be given to one of your peers for evaluation, so you are strongly encouraged to make a copy of it before submission.

Unless otherwise specified, (X, d) is a metric space.

1A. Let $A \subseteq X$, and let $\mathbb{1}_A : X \to \{0, 1\}$ be the indicator function of A. Prove that $\mathbb{1}_A$ is continuous if and only if the set A is both open and closed in X.

2A. Let $A \subseteq X$ be non-empty, and define $d(\cdot, A) : X \to \mathbb{R}$ by $d(x, A) = \inf_{a \in A} d(x, a)$. (For this problem, you may use without proof that compact subsets of $\mathbb{R}$ contain their infimum and supremum.)

i. Prove that $d(\cdot, A)$ is uniformly continuous.

ii. Suppose A is compact, and let $x \in X$. Prove that there exists $a \in A$ such that $d(x, A) = d(x, a)$.

iii. Suppose A is compact. Prove that there exist $a_1, a_2 \in A$ such that $\operatorname{diam}(A) = d(a_1, a_2)$.

3A. Definition: A metric space (X, d) is *arcwise connected* if for all $x, y \in X$, there exists a continuous function $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

i. Let (Y, d_Y) be a metric space, and let $f : X \to Y$ be a continuous surjection. Show that if X is arcwise connected, then so is Y.

ii. Define $f : [0, 1] \to \mathbb{R}$ at $x \neq 0$ by $f(x) = \sin\big(1/x\big)$ and at $x = 0$ by $f(0) = 0$. Prove that the *graph of* f, the set $\Gamma(f) := \big\{\big(x, f(x)\big) \in \mathbb{R}^2 \bigm| x \in [0, 1]\big\}$, is not arcwise connected. (Hint: Can $\Gamma(f)$ be the continuous image of a compact space?)

4A. Consider the metrics d_1 and d_∞ on $\mathbb{R}^2$ defined by $d_1\big((x_1, x_2), (y_1, y_2)\big) = |x_1 - y_1| + |x_2 - y_2|$ and $d_\infty\big((x_1, x_2), (y_1, y_2)\big) = \max(|x_1 - y_1|, |x_2 - y_2|)$. Sketch a picture of $B_1\big((0, 0)\big)$ with respect to both metrics. Show that the metric spaces $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_\infty)$ are isometric with the help of the function $f(x_1, x_2) = \big((x_1 + x_2)/2, (x_1 - x_2)/2\big)$.

5A. Prove that if (X, d) is compact and $f : X \to X$ is an isometry, then f is a homeomorphism. You can follow the outline below if you wish.

i. Show first that it suffices to prove that f is surjective. (This means: supposing f is surjective, finish the details of the proof.)

ii. For $A \subseteq X$ and $r > 0$, denote by $N(A, r)$ the minimal number of open balls of radius r necessary to cover A. Prove that for all $r > 0$, $N(X, r) = N(f(X), r)$.

iii. Prove that if f is not surjective, then there exists $r > 0$ for which $N(f(X), r) < N(X, r)$. (You may use without proof the result of Problem 2Bii on the reverse side.)

Math 5101, NEU, Fall 2018
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Homework #2B

Due October 9, 2018
Peer evaluation due October 18, 2018

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Unless otherwise specified, (X, d) and (Y, d_Y) are metric spaces.

1B. Let $f, g : X \to Y$ be continuous functions, and suppose $A \subseteq X$ is dense. Prove that if for all $a \in A$, $f(a) = g(a)$, then for all $x \in X$, $f(x) = g(x)$.

2B. Let $A \subseteq X$ be non-empty, and define $d(\cdot, A) : X \to \mathbb{R}$ by $d(x, A) = \inf_{a \in A} d(x, a)$. (For this problem, you may use without proof that compact subsets of $\mathbb{R}$ contain their infimum and supremum.)

i. Prove that $d(\cdot, A)$ is uniformly continuous.

ii. Let $x \in X$. Prove that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

iii. Let $A \subseteq X$ be compact and $B \subseteq X$ be closed, and suppose $A \cap B = \emptyset$. Prove that there exists $\delta > 0$ such that for all $a \in A$ and $b \in B$, $d(a, b) > \delta$. (Extra: Can either of the assumptions "A is compact" or "B is closed" be removed?)

3B. Let $A \subseteq X$ be connected, and let $x \in X$. Prove that the set $A \cup \{x\}$ connected if and only if $x \in \overline{A}$. Define $f : [0, 1] \to \mathbb{R}$ at $x \neq 0$ by $f(x) = \sin\big(1/x\big)$ and at $x = 0$ by $f(0) = 0$. Show that the *graph of* f, the set $\Gamma(f) := \big\{\big(x, f(x)\big) \in \mathbb{R}^2 \;\big|\; x \in [0, 1]\big\}$, is connected. (You may use without proof that any continuous image of the interval $(0, 1]$ is connected.)

4B. In this problem, equip the product space $X \times Y$ with the metric $d_{X \times Y}$ defined by $d_{X\times Y}\big((x_1, y_1), (x_2, y_2)\big) = d(x_1, x_2) + d_Y(y_1, y_2)$. Let $f : X \to Y$ be a function, and suppose (X, d) is compact. Prove that f is continuous if and only if the *graph of* f, the set

$$\Gamma(f) := \big\{\big(x, f(x)\big) \in X \times Y \;\big|\; x \in X\big\},$$

is a compact subset of $(X \times Y, d_{X\times Y})$. (Hint: Use the fact that a sequence $\big((x_n, y_n)\big)_{n\in\mathbb{N}} \subseteq X \times Y$ converges if and only if each of the sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $(y_n)_{n\in\mathbb{N}} \subseteq Y$ converge, and employ sequential compactness.)

5B. Prove that if every continuous function $X \to \mathbb{R}$ is uniformly continuous, then X is complete. (Hint: Prove the contrapositive. If $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a Cauchy sequence that does not converge, show that the function $f : X \to \mathbb{R}$ defined by $f(x) = \lim_{n\to\infty} d(x_n, x)$ is continuous and positive. Use this function to furnish a continuous but not uniformly continuous function from X to $\mathbb{R}$.)

Math 5101, NEU, Fall 2018
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Homework #3A

Due October 23, 2018
Peer evaluation due November 1, 2018

Instructions: If you choose to submit the problems on this side, staple this sheet to your work so that this is the first page. Your original work will be given to one of your peers for evaluation, so you are strongly encouraged to make a copy of it before submission.

1A. Let (X, d) and (Y, d_Y) be metric spaces. Let $M > 0$, and let $(f_n : X \to Y)_{n\in\mathbb{N}}$ be a sequence of Lipschitz continuous functions satisfying: for all $n \in \mathbb{N}$ and for all $x, y \in X$, $d_Y\big(f_n(x), f_n(y)\big) \leq Md(x, y)$. Prove or disprove: if $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f : X \to Y$, then f is Lipschitz continuous with the same constant, that is, for all $x, y \in X$, $d_Y\big(f(x), f(y)\big) \leq Md(x, y)$.

2A. Suppose (X, d) is a compact metric space. Prove that if $\mathcal{F} \subseteq C(X, \mathbb{R})$ is equicontinuous and *pointwise bounded* (for all $x \in X$, the set $\{f(x) \mid f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{R}$), then $\mathcal{F}$ is a bounded subset of $C(X, \mathbb{R})$. (Hint: Begin by proving that the map $x \mapsto \sup_{f\in\mathcal{F}} |f(x)|$ is a continuous function from X to $\mathbb{R}$.)

3A. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying: $0 \leq f \leq 1$, for all $r \in \mathbb{R}$, $f(r + 2) = f(r)$, $f|_{[0,1/3]} \equiv 0$, and $f|_{[2/3,1]} \equiv 1$. Define $\gamma : [0, 1] \to [0, 1]^2$ by $\gamma(t) = \big(\sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t)\big)$. Prove that γ is a continuous surjection of $[0, 1]$ onto $[0, 1]^2$. (Hint: Prove that each component of γ is the limit of a uniformly convergent sequence of functions. For further hints, see Rudin, p. 168.)

4A. Suppose that $f \in C([0, 1], \mathbb{R})$ is such that for all integers $n \geq 0$, $\int_0^1 f(x)x^n \, dx = 0$. Prove that f is the constant 0 function. You may use without proof the fact that if $g \in C([0, 1], \mathbb{R})$ is non-negative and non-constant, then $\int_0^1 g(x) \, dx > 0$. (Hint: Show that $\int_0^1 f(x)p(x) \, dx = 0$ for all polynomials $p \in \mathbb{R}[x]$. Then, use Weierstrass approximation to prove that $\int_0^1 f(x)^2 \, dx = 0$.)

5A. (Optional) Let $BC(\mathbb{R}, \mathbb{R})$ be the space of space of bounded, continuous, real-valued functions on $\mathbb{R}$ endowed with the supremum metric: $d(f, g) = \sup_{x\in\mathbb{R}} \big|f(x) - g(x)\big|$. Let $f \in BC(\mathbb{R}, \mathbb{R})$, and for $r \in \mathbb{R}$, define $f_r \in BC(\mathbb{R}, \mathbb{R})$ by $f_r(x) = f(x + r)$. Definition: the function f is *Bohr almost-periodic* if for all $\epsilon > 0$, the set $R_\epsilon := \{r \in \mathbb{R} \mid d(f, f_r) < \epsilon\}$ is *syndetic* (there exists $c > 0$ such that for all $x \in \mathbb{R}$, $R_\epsilon \cap [x, x + c] \neq \emptyset$). Prove that f is Bohr almost-periodic if and only if the subspace $\overline{\{f_r \mid r \in \mathbb{R}\}}$ of $BC(\mathbb{R}, \mathbb{R})$ is compact. You may fill in the details on the following outline if you wish.

i. The subspace $\overline{\{f_r \mid r \in \mathbb{R}\}}$ is complete since it is a closed subset of $BC(\mathbb{R}, \mathbb{R})$. Thus, it suffices to show that f is Bohr a.-p. if and only if $\{f_r \mid r \in \mathbb{R}\}$ is totally bounded.

ii. Prove that for all $r, s \in \mathbb{R}$, $d(f_r, f_s) = d(f_{r-s}, f)$, and use this to prove that if $\{f_r \mid r \in \mathbb{R}\}$ is totally bounded, then f is Bohr a.-p.

iii. Prove that if f is Bohr a.-p., then f is uniformly continuous. Then, use this in conjunction with the first fact from part ii. to prove that $\{f_r \mid r \in \mathbb{R}\}$ is totally bounded.

Math 5101, NEU, Fall 2018
Instructor: Daniel Glasscock Name:

Homework #3B

Due October 23, 2018
Peer evaluation due November 1, 2018

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1B. Use the Arzelà-Ascoli theorem to prove that the subspace

$$K = \left\{ f \in C([0,1],\mathbb{R}) \;\middle|\; f(0) = 0 \text{ and } \big|f(x) - f(y)\big| \leq \sqrt{|x-y|} \right\}$$

of $C([0,1],\mathbb{R})$ is compact. Are $x \mapsto \sqrt{x}$ or $x \mapsto x^2$ in K?

2B. Let (X,d) and (Y,d_Y) be metric spaces, and suppose X is compact. Prove that if $\mathcal{F} \subseteq C(X,Y)$ is equicontinuous, then it is *uniformly equicontinuous*: for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in X$ and all $f \in \mathcal{F}$, if $d(x,y) < \delta$, then $d_Y\big(f(x), f(y)\big) < \epsilon$. (Hint: Follow the proof of the fact that continuous functions on compact spaces are uniformly continuous.)

3B. Let (X,d) and (Y,d_Y) be metric spaces. Let $(f_n : X \to Y)_{n\in\mathbb{N}}$ be a sequence of continuous functions that converges uniformly to $f : X \to Y$, and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence of points that converges to $x \in X$. Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$. Show that uniform convergence is a necessary assumption by giving an example of a pointwise-convergent sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$ on a compact space X and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ that converges to $x \in X$ for which $\lim_{n\to\infty} f_n(x_n) \neq f(x)$.

4B. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq [0,1]$ is *uniformly distributed* if for all $f \in C([0,1],\mathbb{R})$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, dx. \tag{1}$$

Prove *Weyl's criterion*: a sequence $(x_n)_{n\in\mathbb{N}} \subseteq [0,1]$ is uniformly distributed if and only if for all polynomials $f \in \mathbb{R}[x]$, the equality in (1) holds. (Hint: Use Weierstrass approximation and the triangle inequality to bound $\big|N^{-1}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\, dx\big|$ from above.)

5B. (Optional) Suppose (X,d) is a non-empty, compact metric space. Endow $Y := X \times \mathbb{R}$ with the metric $d_Y\big((x_1,r_1),(x_2,r_2)\big) = d(x_1,x_2) + |r_1 - r_2|$. Let $\mathcal{F}(Y)$ be the set of non-empty, compact subsets of Y endowed with the Hausdorff metric d_H, defined for $F, H \in \mathcal{F}(Y)$ by

$$d_H\big(F,H\big) = \inf\{\delta \geq 0 \mid F \subseteq [H]_\delta \text{ and } H \subseteq [F]_\delta\},$$

where $[F]_\delta = \{y \in Y \mid \exists f \in F,\ d_Y(y,f) \leq \delta\}$ is the closed δ-neighborhood of F. We showed in Homework #2, problem 4B that $f : X \to \mathbb{R}$ is continuous if and only if its graph, $\Gamma(f) \subseteq Y$, is compact, that is, $\Gamma(f) \in \mathcal{F}(Y)$. Let $(f_n)_{n\in\mathbb{N}} \subseteq C(X,\mathbb{R})$ and $f \in C(X,\mathbb{R})$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f if and only if $\big(\Gamma(f_n)\big)_{n\in\mathbb{N}} \subseteq \mathcal{F}(Y)$ converges to $\Gamma(f) \in \mathcal{F}(Y)$ with respect to the Hausdorff metric. (Hint: the "only if" direction is straightforward from the definitions. For the "if" direction, make use of the fact that f is uniformly continuous.)

Math 5101, NEU, Fall 2018
Instructor: Daniel Glasscock Name:

Homework #4A

Due November 6, 2018
Peer evaluation due November 15, 2018

Instructions: If you choose to submit the problems on this side, staple this sheet to your work so that this is the first page. Your original work will be given to one of your peers for evaluation, so you are strongly encouraged to make a copy of it before submission.

Unless otherwise specified, all function spaces, such as $C^1(\mathbb{R})$, consist of *real-valued* functions.

1A. Let $a, c \in \mathbb{R}$, $c > 0$. Prove that $x \mapsto |x|^a$ is continuously differentiable on $\mathbb{R}$ if and only if $a > 1$. Then, define $f : [-1, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} |x|^a \sin\left(|x|^{-c}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, and prove that:

i. f is continuous if and only if $a > 0$;

ii. f is continuously differentiable if and only if $a > 1 + c$; and

iii. f is twice continuously differentiable if and only if $a > 2 + 2c$.

2A. Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$.

i. Suppose f is differentiable at x. Prove that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h}.$$

ii. Suppose f is twice differentiable at x. Prove that

$$f''(x) = \lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}.$$

(Hint: On part ii., consider appealing to L'Hospital's theorem.)

3A. Define

$$BC^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; f \in BC(\mathbb{R}) \cap C^1(\mathbb{R}) \text{ and } f' \in BC(\mathbb{R}) \right\},$$

and for $f, g \in BC^1(\mathbb{R})$, define $d(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)| + \sup_{x \in \mathbb{R}} |f'(x) - g'(x)|$. Prove that $(BC^1(\mathbb{R}), d)$ is a metric space. Prove that the derivative operator $\cdot' : BC^1(\mathbb{R}) \to BC(\mathbb{R})$, where $BC(\mathbb{R})$ is endowed with the uniform metric, is continuous.

4A. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$, and suppose that there exist $A, C > 0$ such that $|f| \leq A$ and $|f''| \leq C$. Prove that $|f'| \leq A + C$. (Hint: Fix $x \in \mathbb{R}$ and $h > 0$, and use Taylor's theorem to write $f'(x)$ in terms of, among other things, $f(x + 2h)$.)

5A. Let $f \in C^1(\mathbb{R})$, and suppose $\lim_{x \to \infty} f'(x)$ exists and is equal to $L \in \overline{\mathbb{R}}$ (in the extended real numbers). Show that $\lim_{x \to \infty} f(x)/x = L$. (This is a special case of L'Hospital's theorem, so do not appeal to that theorem without re-proving it first.) (Hint: Use the mean value theorem.)

Math 5101, NEU, Fall 2018
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Homework #4B

Due November 6, 2018
Peer evaluation due November 15, 2018

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Unless otherwise specified, all function spaces, such as $C^1(\mathbb{R})$, consist of *real-valued* functions.

1B. Let $f : \mathbb{R} \to \mathbb{R}$, and suppose that for all $x, y \in \mathbb{R}$, $\big|f(x) - f(y)\big| \leq (x-y)^2$. Prove that f is constant.

2B. For $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{x}{1+nx^2}.$$

Prove that $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuously differentiable functions that converges uniformly to a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ but that the sequence $(f_n')_{n\in\mathbb{N}}$ does not even converge pointwise to f'.

3B. Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable. Prove that f is *uniformly differentiable on* $[0,1]$: for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $0 \leq x \leq y \leq z \leq 1$ with $0 < |x-z| < \delta$,

$$\left| \frac{f(x) - f(z)}{x - z} - f'(y) \right| < \epsilon.$$

4B. Let $f \in C^2(\mathbb{R})$, and suppose that $f \geq 0$ and $f'' \leq 1$. Prove that $(f')^2 \leq 2f$. (Hint: Fix $x, y \in \mathbb{R}$, and use Taylor's theorem to approximate $f(x+y)$. Use the given bounds along with what you know about the discriminant of a quadratic polynomial.)

5B. Let $f : (1,\infty) \to \mathbb{R}$ be differentiable, and define $g : (1,\infty) \to \mathbb{R}$ by $g(x) = f(x)/x$. Prove that if there exist $A > 0$ such that for all $x > 1$, $|f'(x)| \leq Ax$, then g is uniformly continuous. You may use the outline below if you wish.

i. Show that it suffices to prove that g is differentiable and that g' is bounded.

ii. Show that f is bounded on $(1,2]$ using the mean value theorem, and conclude that g' is bounded on $(1,2]$.

iii. Show that g' is bounded on $[2,\infty)$ using the mean value theorem.

Math 5101, NEU, Fall 2018
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Homework #5A

Due November 20, 2018
Peer evaluation due December 4, 2018

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Unless otherwise specified, all function spaces, such as $C^1(\mathbb{R})$, consist of *real-valued* functions.

1A. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if x is irrational and $f(x) = 1/m$ if $x = n/m$, where $n, m \in \mathbb{Z}$, $m \geq 1$, are relatively prime. Prove from the definition of the Riemann integral that f is Riemann integrable on the interval $[0,1]$. Prove that the related function $\mathbb{1}_{\mathbb{Q}}$ is not Riemann integrable on the interval $[0,1]$.

2A. Prove that if $g \in C([0,1])$ is non-negative and non-constant, then $\displaystyle\int_0^1 g(x)\ dx > 0$. Combine this with the Bunyakovsky-Cauchy-Schwarz inequality to show that

$$d(f,g) = \left(\int_0^1 \big(f(x) - g(x)\big)^2\ dx \right)^{1/2}$$

is a metric on $C([0,1])$. Show that the metric space $\big(C([0,1]), d\big)$ is not complete.

3A. Let $f \in C^1([0,1])$. Prove that

$$\lim_{N\to\infty} \left(\sum_{n=1}^{N} f\left(\frac{n}{N}\right) - N \int_0^1 f(x)\ dx \right) = \frac{f(1) - f(0)}{2}.$$

(Hint: Denote by S_N the expression inside the parenthesis on the left hand side. Write $S_N = N \sum_n \int \big(f(n/N) - f(x)\big)\ dx$ with the correct limits of summation and integration. Compare S_N to $\sum_n f'(n/N)/2N$ using the uniform continuity of f', and recognize a Riemann sum.)

4A. Definition: the support of $f \in C(\mathbb{R})$ is the set $\text{supp}(f) = \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$. Definition: the sumset of $A, B \subseteq \mathbb{R}$ is $A + B = \{a + b \mid a \in A,\ b \in B\}$. Let $f, g \in C(\mathbb{R})$, and suppose f has compact support. Prove that $\text{supp}(f * g) \subseteq \text{supp}(f) + \text{supp}(g)$, and give an example showing that the inclusion need not be an equality. Conclude that if both f and g have compact support, then $f * g$ has compact support.

5A. Suppose $K \in C(\mathbb{R}^2)$ is such that for all $x, y, z \in \mathbb{R}$, $K(x, y+z) = K(x,y)K(x,z)$. For $f \in C(\mathbb{R})$ with compact support, define the function $T_K f : \mathbb{R} \to \mathbb{R}$ by

$$\big(T_K f\big)(x) = \int_{-\infty}^{\infty} K(x,y) f(y)\ dy.$$

Prove that if $f, g \in C(\mathbb{R})$ have compact support, then $T_K(f * g) = (T_K f)(T_K g)$.

Math 5101, NEU, Fall 2018
Instructor: Daniel Glasscock Name:

Homework #5B

Due November 20, 2018
Peer evaluation due December 4, 2018

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Unless otherwise specified, all function spaces, such as $C^1(\mathbb{R})$, consist of *real-valued* functions.

1B. Prove from the definition of the Riemann integral that if $f : [0,1] \to \mathbb{R}$ is Riemann integrable, then $|f| : [0,1] \to \mathbb{R}$ is Riemann integrable.

2B. Let $f, g \in C([a,b])$ with $g \geq 0$. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\ dx = f(c) \int_a^b g(x)\ dx.$$

(Hint: You might find it helpful to use the fact that $f([a,b])$ is an interval in $\mathbb{R}$.)

3B. Let $f \in C([0,1])$ and $g \in C(\mathbb{R})$. Prove that if g is periodic with period 1 (for all $x \in \mathbb{R}$, $g(x+1) = g(x)$), then

$$\lim_{N\to\infty} \int_0^1 f(x)g(Nx)\ dx = \int_0^1 f(x)\ dx \int_0^1 g(x)\ dx.$$

(This is a special case of the Riemann-Lebesgue lemma.) (Hint: Show that we can assume without loss of generality that $g \geq 0$. Using the periodicity of g, write $\int_0^1 f(x)g(Nx)\ dx = N^{-1}\sum_n \int f\big((n+y)/N\big)g(y)\ dy$ with the correct limits of summation and integration. Then, apply the result of Problem 2B to each integral in the sum and recognize a Riemann sum.)

4B. Prove that if $f \in C(\mathbb{R})$ has compact support and $g \in B(\mathbb{R})$ is Riemann integrable on every compact interval, then $f * g \in C(\mathbb{R})$. (Be careful: we do not assume that $\int_{-\infty}^{\infty} |g(x)|\ dx < \infty$.) Give an example of $f, g \in C(\mathbb{R})$ with f uniformly continuous such that $f * g$ is not uniformly continuous on $\mathbb{R}$.

5B. Set $X = [0,1]$, and let $K \in C(X^2)$. For $f \in C(X)$, define the function $T_K f : X \to \mathbb{R}$ by

$$\big(T_K f\big)(x) = \int_0^1 K(x,y)f(y)\ dy.$$

Prove that if $f \in C(X)$, then $T_K f \in C(X)$. Prove that the map $T_K : C(X) \to C(X)$ is continuous.

Math 5101, NEU, Fall 2018
Instructor: Daniel Glasscock   Name:

Homework #6A

Due December 11, 2018

Instructions: If you choose to submit the problems on this side, staple this sheet to your work so that this is the first page. Your original work will be given to one of your peers for evaluation, so you are strongly encouraged to make a copy of it before submission.

1A. Consider the following two norms on $\mathbb{R}^{n\times n}$: the Euclidean norm $\|A\|_E = (\sum_{i,j=1}^n A_{i,j}^2)^{1/2}$ induced from $\mathbb{R}^{n^2}$ and the operator norm $\|A\|_{\mathrm{op}} = \sup_{|x|\leq 1} |Ax|$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Prove that for all $A \in \mathbb{R}^{n\times n}$,

$$\|A\|_{\mathrm{op}} \leq \|A\|_E \leq \sqrt{n}\|A\|_{\mathrm{op}}.$$

(This shows that the Euclidean and operator norms on $\mathbb{R}^{n\times n}$ are *equivalent*. In fact, *all* norms on a finite dimensional vector space are equivalent.)

2A. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}^n$ and f attains a local maximum at x_0, then $f'(x_0) = 0$. (Hint: Reduce this to the analogous statement in one dimension by considering directional derivatives.)

3A. Definition: a matrix $B \in \mathbb{R}^{n\times n}$ is a *square root* of the matrix $A \in \mathbb{R}^{n\times n}$ if $B^2 = A$. Give an example of a matrix with no square root. Equip $\mathbb{R}^{n\times n}$ with the Euclidean metric induced from $\mathbb{R}^{n^2}$. Prove that there exists $\delta > 0$ such that all matrices in $B_\delta(I)$ have a square root. (Hint: Prove that $f : A \mapsto A^2$ is continuously differentiable on $\mathbb{R}^{n\times n}$. Compute $f'(I)$, and appeal to the inverse function theorem.)

4A. Consider the system of equations

$$\begin{cases} \sin(x+y) + z^2 = 1 \\ y^2 + xz^2 = \pi^2 \end{cases}.$$

Show that y and z can be solved as functions of x in a neighborhood of the point $x = 0$: there exists an open interval $I \subseteq \mathbb{R}$ containing 0 and functions $g, h \in C^1(I, \mathbb{R})$ so that for all $x \in I$, $(x, y, z) = (x, g(x), h(x))$ is a solution to the system.

5A. Definition: a curve $\gamma : [a, b] \to \mathbb{R}^n$ is a *streamline* for the vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if for all $t \in [a, b]$, $\gamma'(t) = F(\gamma(t))$. Let $F(x, y) = (x^2 - y, y^2)$. Prove that there exists a closed interval $I \subseteq \mathbb{R}$ containing 0 and a streamline $\gamma : I \to \mathbb{R}$ for F with $\gamma(0) = (-1, -1)$. Sketch the vector field F in the window $[-2, 2]^2$, and sketch (part of) the streamline passing through $(-1, -1)$.

Math 5101, NEU, Fall 2018
Instructor: Daniel Glasscock Name:

Homework #6B

Due December 11, 2018

Instructions: If you choose to submit the problems on this side, staple this sheet to your work so that this is the first page. Your original work will be given to one of your peers for evaluation, so you are strongly encouraged to make a copy of it before submission.

1B. Equip $\mathbb{R}^{n\times n}$ with the operator norm $\|\cdot\|$ induced from the Euclidean norm on $\mathbb{R}^n$. Let $A, B \in \mathbb{R}^{n\times n}$. In this problem, you may use without proof the fact that $\|AB\| \leq \|A\|\|B\|$ and that matrix multiplication $(A, B) \mapsto AB$ is continuous on $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$.

i. Prove that if $\|A\| < 1$, then $\sum_{n=0}^{\infty} A^n = (I - A)^{-1}$. Conclude that if $\|A + I\| < 1$, then A is invertible.

ii. Prove that if B is invertible and $\|B^{-1}A\| < 1$, then $A + B$ is invertible. (Hint: Factor out B from $A + B$ and apply part i.)

iii. Prove that the subset of $\mathbb{R}^{n\times n}$ consisting of invertible matrices is open.

2B. Let $E \subseteq \mathbb{R}^n$ be open and connected, and let $f : E \to \mathbb{R}^m$ be differentiable. Show that if for all $x \in E$, $f'(x) = 0$, then f is constant. (Hint: Show using Theorem 9.19 from Rudin that f is *locally constant*: for all $x \in E$, there exists $\delta > 0$ such that for all $y \in B_\delta(x)$, $f(y) = f(x)$. Fix $c \in f(E)$, and show that the set $\{x \in E \mid f(x) = c\}$ is non-empty, open, and closed. Finally, invoke the connectedness of E.)

3B. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 2x^2 \sin(x^{-1})$ when $x \neq 0$ and $f(x) = 0$ when $x = 0$. Prove that $f'(0) = 1$, that f' is bounded on $[-1, 1]$, and that f is not injective on any open neighborhood of the origin. Conclude that the assumption on the continuity of the derivative in the statement of the inverse function theorem is necessary.

4B. Consider the equation $x^2y + e^x + z = 0$ where $x, y, z \in \mathbb{R}$. Show that x can be solved as a function of y and z in a neighborhood of the point $(y, z) = (1, -1)$: there exists an open set $E \subseteq \mathbb{R}^2$ containing $(1, -1)$ and a function $g \in C^1(E, \mathbb{R})$ so that for all $(y, z) \in E$, $(x, y, z) = (g(y, z), y, z)$ is a solution to the equation. Find $g'(1, -1)$.

5B. Prove that there exists a unique $f \in C^2([-1, 1], \mathbb{R})$ that solves the initial value problem $f''(x) = x^2 f'(x) + f(x)\sin(x)$, $f(0) = 1$, $f'(0) = 2$.