What is the crank of a partition?

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These notes complement a talk given for the What is ... ? *seminar at the Ohio State University.*

Introduction

The crank of an integer partition is an integer statistic which yields a combinatorial explanation of Ramanujan's three famous congruences for the partition function. The existence of such a statistic was conjectured by Freeman Dyson in 1944 and realized 44 years later by George Andrews and Frank Gravan. Since then, groundbreaking work by Ken Ono, Karl Mahlburg, and others tells us that not only are there infinitely many congruences for the partition function, but infinitely many of them are explained by the crank.

In this note, we will introduce the partition function, define the crank of a partition, and show how the crank underlies congruences of the partition function. Along the way, we will recount the fascinating history behind the crank and its older brother, the rank. This exposition ends with the recent work of Karl Mahlburg on crank function congruences. Andrews and Ono [3] have a good quick survey from which to begin.

The partition function

The *partition function* $p(n)$ counts the number of distinct partitions of a positive integer n, where a *partition* of n is a way of writing n as a sum of positive integers. For example, $p(4) = 5$ since 4 may be written as the sum of positive integers in 5 essentially different ways: 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, and $1 + 1 + 1 + 1$. The numbers comprising a partition are called *parts* of the partition, thus $2 + 1 + 1$ has three parts, two of which are odd, and is not composed of distinct parts (since the 1 repeats).

The partition function is a central object in combinatorics and number theory. To get a feeling for it, let's revisit three landmarks in the history of $p(n)$.

1. Leonhard Euler was perhaps the first to study the partition function seriously. He showed [9] in 1748 that
$$p(0) + p(1)x + p(2)x^2 + \cdots = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \cdots ,$$
that is, the generating function of the sequence $\big\{p(n)\big\}_{n=0}^{\infty}$ is (formally) the infinite product of the rational functions $(1 - x^k)^{-1}$. (We adopt the convention here that $p(0) = 1$.) Euler used this generating function to deduce a recurrence relation for $p(n)$ and prove interesting combinatorial facts about partitions. For example, he showed that the number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts.

2. Srinivasa Ramanujan and Godfrey Harold Hardy [10] in 1918, and independently James Victor Uspensky [16] in 1920, showed that
$$p(n) \approx \frac{1}{4\sqrt{3}\, n} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty.$$
(Paul Erdős [8] later gave an elementary proof of this fact.) An explicit formula for $p(n)$ in the form of an infinite, convergent series was achieved by Hans Rademacher [14] in 1938. It is an interesting exercise to devise the most elementary argument you can showing that the growth of $p(n)$ is super-polynomial but at most exponential.

3. Ramanujan [15] in 1919 stated the following congruences for $p(n)$: for all $k \geq 0$,
$$p(5k + 4) \equiv 0 \pmod 5, \tag{1}$$
$$p(7k + 5) \equiv 0 \pmod 7, \tag{2}$$
$$p(11k + 6) \equiv 0 \pmod{11}. \tag{3}$$

> The first congruence, for example, gives that the number of partitions of any number ending in the digit 4 or 9 is a multiple of 5! Ramanujan proved the first two congruences in his paper, and the proof of the third was extracted by Hardy from Ramanujan's notebooks after Ramanujan's death in 1920. Recent work by Ken Ono and others has revolutionized what we know about partition function congruences; we'll revisit this later on in the talk.

The crank of a partition is related to congruences of the partition function, so we will follow this line in more depth. Before defining the crank, it is worthwhile to introduce its older brother, the rank.

The rank of a partition

Since $p(5k+4)$ is always divisible by 5, one may wonder whether there is a natural way split the partitions of $5k+4$ up into 5 groups of equal size. In searching for this sort of combinatorial explanation to Ramanujan's congruences, Freeman Dyson [7] in 1944, then an undergraduate at the University of Cambridge, defined the *rank* of a partition π to be

$$\operatorname{rank}(\pi) = \big(\text{largest part of } \pi\big) - \big(\#\text{ of parts of } \pi\big).$$

For example, $\operatorname{rank}(5+3+2+1) = 1$ since 5 is the largest of the 4 parts (of this partition of 11). Based on empirical evidence (see Tables 1, 2, 4, and 6), Dyson conjectured that the rank modulo 5 naturally splits the partitions of $5k+4$ into 5 groups of equal size. He conjectured additionally the analogous behavior for 7 with the partitions of $7k+5$. The rank, in this sense, would provide a nice combinatorial explanation of (1) and (2), Ramanujan's first and second congruences.

To facilitate the discussion, define

$$\mathcal{N}(m,q,n) = \text{number of partitions of } n \text{ with rank equal to } m \text{ modulo } q.$$

In this notation, Dyson's conjectures become: for all $k \geq 0$,

$$\mathcal{N}(0,5,5k+4) = \mathcal{N}(1,5,5k+4) = \cdots = \mathcal{N}(4,5,5k+4), \text{ and}$$
$$\mathcal{N}(0,7,7k+5) = \mathcal{N}(1,7,7k+5) = \cdots = \mathcal{N}(6,7,7k+5).$$

Ten years later, Arthur Oliver Lonsdale Atkin and Peter Swinnerton-Dyer [5] proved these conjectures of Dyson, establishing the rank as a partition statistic which satisfactorily describes the first two of Ramanujan's congruences combinatorially.

Mysteriously, the rank modulo 11 completely fails to provide any sort of explanation for (3), Ramanujan's third congruence. This is apparent already for partitions of 6, the first $\{11k+6\}$; see Table 3. Dyson was aware of this (painfully, perhaps), but he was optimistic about the approach. He conjectured the existence of another partition statistic, the *crank* of a partition, which, modulo 11, was to explain Ramanujan's third congruence just as the rank explains the first two. It is worth revisiting Dyson's original text on this point:

> *The regularity of this table is of precisely the same character as the regularity of the previous one. One is thus led irresistibly to the conclusion that there must be some analogue modulo 11 to the relations (26).*
>
> *I hold in fact:*
>
> *That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition, and denote by $M(m,q,n)$ the number of partitions of n whose crank is congruent to m modulo q;*
>
> *that $M(m,q,n) = M(q-m,q,n)$;*
>
> *that $M(0,11,11n+6) = M(1,11,11n+6) = \cdots = M(4,11,11n+6)$;*
>
> *that numerous other relations exist analogous to (12) – (19) ...*

Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe the "crank" is unique among the arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!

The planet Vulcan was hypothesized by the 19th century French mathematician Urbain Le Verrier to explain peculiarities in the orbit of Mercury. Unlike the planet Vulcan, whose existence never panned out, Dyson's crank was realized more than 40 years later.

The crank of a partition

In 1988, George Andrews and Frank Garvan [2] finally realized Dyson's crank statistic. For a partition π, let $\mathbb{1}(\pi)$ be the number of 1's in π. They defined the *crank* of the partition π to be

$$\operatorname{crank}(\pi) = \begin{cases} \text{largest part of } \pi & \text{if } \mathbb{1}(\pi) = 0 \\ \big(\#\text{ of parts greater than } \mathbb{1}(\pi)\big) - \mathbb{1}(\pi) & \text{if } \mathbb{1}(\pi) \geq 1 \end{cases}.$$

For example, $\operatorname{crank}(4+2+1+1) = -1$ since there is 1 part (namely, 4) greater than the number of 1's (there are 2), and $\operatorname{crank}(2+2+2) = 2$ since there are no 1's in this partition and 2 is its largest part.

In the same way as the rank gives combinatorial justification to Ramanujan's first two congruences, the crank explains Ramanujan's third: Andrews and Garvan proved that the crank modulo 11 splits the partitions of $11k+6$ into 11 groups of equal size! In fact, they showed that the crank modulo 5 and 7 splits the partitions of $5k+4$ and $7k+5$ into 5 and 7 groups of equal size, respectively. In other words, the crank explains combinatorially all three of Ramanujan's congruences!

Let's introduce the aogue to $\mathcal{N}$ for clarity. Define

$$\mathcal{M}(m, q, n) = \text{number of partitions of } n \text{ with crank equal to } m \text{ modulo } q.$$

In this notation, Andrews and Garvan's theorem is that for all $k \geq 0$,

$$\begin{gathered} \mathcal{M}(0,5,5k+4) = \mathcal{M}(1,5,5k+4) = \cdots = \mathcal{M}(4,5,5k+4), \\ \mathcal{M}(0,7,7k+5) = \mathcal{M}(1,7,7k+5) = \cdots = \mathcal{M}(6,7,7k+5), \text{ and} \\ \mathcal{M}(0,11,11k+6) = \mathcal{M}(1,11,11k+6) = \cdots = \mathcal{M}(10,11,11k+6). \end{gathered}$$

To see the crank in action, consider Tables 1, 2, 3, and 5. For partitions of 4, 5, and 6, the crank modulo 5, 7, and 11, respectively, is unique; that is, the crank induces an equinumerous splitting of the partitions in which each group has one partition. Comparing Tables 4 and 5, we see that the crank and rank induce *different* equinumerous splittings of the partitions of 9.

Fairy-tale endings are not common in mathematics, and this story is no exception. Indeed, there are many more congruences for the partition function far beyond Ramanujan's original three, each warranting a similar combinatorial explanation. It turns out that the crank does continue to provide explanations for infinitely many of these congruences, albeit in a slightly weaker manner.

Further congruences and behavior of the crank

The partition function satisfies additional congruences similar to the original ones of Ramanujan. In fact, Ramanujan conjectured, and it was later shown, that such congruences exist modulo arbitrary powers of 5, 7, and 11. In 1967, Atkin and J. N. O'Brien [4] discovered further congruences; for example, for all $k \geq 0$,

$$p\big(17303k + 237\big) \equiv 0 \pmod{13}.$$

In groundbreaking work in the year 2000, Ken Ono [13] showed that there are infinitely many families of such congruences.

Theorem 1 (Ono)**.** *Let $q \geq 5$ be prime. There exist infinitely many non-nested arithmetic progressions $\{Ak+B\}_{k=0}^{\infty}$ such that for all $k \geq 0$,*

$$p\big(Ak+B\big) \equiv 0 \pmod{q}.$$

Some specific examples of such higher congruences (from [13], [3], [12], respectively) include

$$\begin{aligned} p\big(157525693k+111247\big) &\equiv 0 \pmod{13}, \\ p\big(48037937k+1122838\big) &\equiv 0 \pmod{17}, \\ p\big(1977147619k+815655\big) &\equiv 0 \pmod{19}. \end{aligned}$$

Ahlgren and Ono [1] went on to show that such congruences exist for the partition function for any modulus q coprime to 6!

In light of our previous discussion, one is led naturally to ask whether or not the crank provides the same sort of combinatorial justification for these higher congruences as it does for Ramanujan's congruences. Addressing this question, Karl Mahlburg [11, 12] in 2005, then a Ph.D. student of Ono's, showed that the crank functions $\mathcal{M}(m,q,n)$ satisfy congruence relations similar to those of the partition function. These congruences for the crank underlie congruences for the partition function!

To make this precise, note first that by the definition of $\mathcal{M}(m,q,n)$, by summing over all residue classes modulo q,

$$p(n) = \sum_{m=0}^{q-1} \mathcal{M}(m,q,n). \tag{4}$$

When $q = 5$, 7, or 11 and $n = 5k+4$, $7k+5$, or $11k+6$, respectively, Andrews and Garvan's theorem tells us that all q factors in this sum are equal, yielding immediately that $p(n)$ is divisible by q. Mahlburg reached the same conclusion on the divisibility of $p(n)$ by showing that all q factors in this sum are *divisible* by q.

Theorem 2 (Mahlburg)**.** *Let $q \geq 5$ be prime. There exists infinitely many non-nested arithmetic progressions $\{Ak+B\}_{k=0}^{\infty}$ such that for all $k \geq 0$ and for all $0 \leq m \leq q-1$,*

$$\mathcal{M}(m,q,Ak+B) \equiv 0 \pmod{q}.$$

In other words, Mahlburg's theorem shows that the crank yields a combinatorial explanation for infinitely many families of congruences for the partition function!

To wrap up, let's address three of the many questions one is led naturally to ask.

First, does the crank yield an *equinumerous* splitting of the partitions of n (along arithmetic progressions) modulo q for any q besides 5, 7, and 11 (as it does for Ramanujan's congruences)? Mahlburg writes in his thesis [12] in 2005 that the behavior of the crank modulo these small primes is atypical:

> *The crank functions modulo N differ from $p(n)/N$ by a meromorphic modular form, and it is reasonable to expect that with the exception of finitely many small primes and arithmetic progressions (which will perhaps turn out to include only those for Ramanujan's congruences), this difference is never zero. As a consequence, Theorem [2] would then be the best possible result, in the sense that there are no infinite families of partition congruences that are grouped into equal classes by the crank.*

According to Mahlburg (personal communication, July 2014), this question is still unresolved.

Second, what about behavior of the rank for these higher congruences? In 2009, Kathrin Bringmann [6] showed that Dyson's rank functions also satisfy infinite families of congruences.

Theorem 3 (Bringmann)**.** *Let $q \geq 5$ be prime. There exists infinitely many non-nested arithmetic progressions $\{Ak+B\}_{k=0}^{\infty}$ such that for all $k \geq 0$ and for all $0 \leq m \leq q-1$,*

$$\mathcal{N}(m,q,Ak+B) \equiv 0 \pmod{q}.$$

Writing the analogue of (4) for the rank functions, Bringmann's theorem gives immediately that the rank underlies infinitely many families of congruences for the partition function, just like the crank!

Finally, which congruences are explained by the rank and crank? Mahlburg and Bringmann's theorems give only existence and don't provide, as stated, concrete examples. If there are congruences not explained by the rank or crank, are there other partition statistics which yield in a similar fashion combinatorial explanations to large families of congruences of the partition function?

This seems to be a nice, motivational, open-ended place to end these notes.

Tables

Table 1: Partitions of 4

partition	rank	rank (mod 5)	crank	crank (mod 5)
4	3	3	4	4
$3+1$	1	1	0	0
$2+2$	0	0	2	2
$2+1+1$	-1	4	-2	3
$1+1+1+1$	-3	2	-4	1

Table 2: Partitions of 5

partition	rank	rank (mod 7)	crank	crank (mod 7)
5	4	4	5	5
$4+1$	2	2	0	0
$3+2$	1	1	3	3
$3+1+1$	0	0	-1	6
$2+2+1$	-1	6	1	1
$2+1+1+1$	-2	5	-3	4
$1+1+1+1+1$	-4	3	-5	2

Table 3: Partitions of 6

partition	rank	rank (mod 11)	crank	crank (mod 11)
6	5	5	6	6
$5+1$	3	3	0	0
$4+2$	2	2	4	4
$4+1+1$	1	1	-1	10
$3+3$	1	1	3	3
$3+2+1$	0	0	1	1
$3+1+1+1$	-1	10	-3	8
$2+2+2$	-1	10	2	2
$2+2+1+1$	-2	9	-2	9
$2+1+1+1+1$	-3	8	-4	7
$1+1+1+1+1+1$	-5	6	-6	5

Table 4: Partitions of 9 grouped by the rank modulo 5

rank $\equiv 0$	rank $\equiv 1$	rank $\equiv 2$	rank $\equiv 3$	rank $\equiv 4$
$7+2$	$8+1$	$6+1+1+1$	9	$7+1+1$
$5+1+1+1+1$	$5+2+1+1$	$5+3+1$	$6+2+1$	$6+3$
$4+3+1+1$	$4+4+1$	$5+2+2$	$5+4$	$4+2+1+1+1$
$4+2+2+1$	$4+3+2$	$3+2+1+1+1+1$	$3+3+1+1+1$	$3+3+2+1$
$3+3+3$	$3+1+1+1+1+1+1$	$2+2+2+2+1$	$4+1+1+1+1+1$	$3+2+2+2$
$2+2+1+1+1+1+1$	$2+2+2+1+1+1$	$1+1+1+1+1+1+1+1+1$	$3+2+2+1+1$	$2+1+1+1+1+1+1+1$

Table 5: Partitions of 9 grouped by the crank modulo 5

crank $\equiv 0$	crank $\equiv 1$	crank $\equiv 2$	crank $\equiv 3$	crank $\equiv 4$
$8+1$	$6+3$	$7+2$	$6+1+1+1$	9
$5+4$	$6+2+1$	$5+1+1+1+1$	$4+2+1+1+1$	$7+1+1$
$5+2+2$	$5+3+1$	$4+2+2+1$	$3+3+3$	$5+2+1+1$
$4+3+1+1$	$4+4+1$	$3+3+2+1$	$3+2+2+2$	$4+3+2$
$4+1+1+1+1+1$	$3+2+1+1+1+1$	$3+3+1+1+1$	$2+2+2+2+1$	$3+2+2+1+1$
$2+2+1+1+1+1+1$	$1+1+1+1+1+1+1+1+1$	$2+2+2+1+1+1$	$2+1+1+1+1+1+1+1$	$3+1+1+1+1+1+1$

Table 6: Table in Dyson's *Some guesses in the theory of partitions*, [7]
$a=\mathcal{N}(0,5,n)-\mathcal{N}(1,5,n)$, $b=\mathcal{N}(1,5,n)-\mathcal{N}(2,5,n)$

n	a	b	n	a	b	n	a	b	n	a	b	n	a	b
1	1	0	2	0	1	3	0	−1	4	0	0	5	0	1
6	1	0	7	0	0	8	−1	−1	9	0	0	10	0	1
11	1	0	12	0	1	13	0	−2	14	0	0	15	−1	1
16	1	0	17	0	1	18	−1	−2	19	0	0	20	0	1
21	2	0	22	0	1	23	−1	−2	24	0	0	25	−1	2
26	1	0	27	0	0	28	−1	−3	29	0	0	30	0	2
31	2	0	32	0	2	33	−1	−3	34	0	0	35	−1	2
36	2	0	37	0	1	38	−2	−4	39	0	0	40	−1	2
41	3	0	42	0	2	43	−1	−4	44	0	0	45	−2	3
46	3	0	47	0	2	48	−2	−5	49	0	0	50	0	3

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