Topology and Convergence

by: *Daniel Glasscock*, May 2012

These notes grew out of a talk I gave at The Ohio State University. The primary reference is [1]. A possible error in the proof of Theorem 1 in [1] is corrected here. (Updated: May 15, 2012.)

Warning: It was brought to my attention that some set-theoretic care is required when making statements about all subnets of a net. Because of the definition of a subnet, the collection of all subnets of a net may too large to be a set. One runs the risk of a "set of all sets." No care is taken in these notes to remedy this.

1. Introduction

We are frequently in the following situation:

> Let A, B be topological spaces and X be a subset of functions from A to B. Endow X with the topology of pointwise convergence; i.e. the topology in which $f_n \longrightarrow f$ if and only if for all $a \in A$, $f_n(a) \longrightarrow f(a)$.

The ubiquitous phrase "topology of pointwise convergence" seems to suggest two things: there is a topology determined by the notion of pointwise convergence, and this topology is the unique topology which yields this convergence on X. (There is at least one familiar topology which has this property: X as a subset of B^A inherits a topology from the product topology.)

Given a topology τ on a set X, let $\mathcal{C}_\tau$ be the collection of all pairs (N, x) where N is a net in X converging to the point $x \in X$ with respect to τ. (See section 3 for definitions and notation regarding nets.) We are led naturally to ask:

- To what extent does $\mathcal{C}_\tau$ determine the topology τ?
- Is there another topology δ on X such that $\mathcal{C}_\delta = \mathcal{C}_\tau$?
- If $\mathcal{C}$ is a collection of pairs of nets and points in X, is there a topology δ on X such that $\mathcal{C}_\delta = \mathcal{C}$?

Given a collection $\mathcal{C}$ of pairs of nets and points in X, there are some immediate necessary conditions on $\mathcal{C}$ if we hope to find a topology δ on X such that $\mathcal{C}_\delta = \mathcal{C}$. For example, constant nets converge in any topology, so pairs of constant nets with their limit points must exist in $\mathcal{C}$.

Definition A *convergence class* $\mathcal{C}$ *on a set* X is a collection of pairs $\mathcal{C} = \{(N, x) \mid N \text{ net in } X, x \in X\}$ with the following properties:

i) If N is a constant net at x, then $(N, x) \in \mathcal{C}$.

ii) If $(N, x) \in \mathcal{C}$, then for all subnets N' of N, $(N', x) \in \mathcal{C}$.

iii) If N is a net and x is a point such that for all subnets N' of N, there is a subnet N'' of N' such that $(N'', x) \in \mathcal{C}$, then $(N, x) \in \mathcal{C}$.

iv) If $((N, D), x) \in \mathcal{C}$ and for each $d \in D$, $(M_d, N(d)) \in \mathcal{C}$, then $(L, x) \in \mathcal{C}$ where L is the diagonal net of N and the M_d's.

Given a net N in X, we say N *converges* $(\mathcal{C})$ *to* x when $(N, x) \in \mathcal{C}$.

That conditions i) – iv) on $\mathcal{C}$ are necessary – that $\mathcal{C}_\tau$ is a convergence class – is the content of Lemmas 2 – 5. The striking fact is that they are sufficient to generate a topology δ on X with exactly the specified convergence class $\mathcal{C}$. The goal of the remainder of these notes is the proof of the following theorem.

Theorem 1 *Let X be a set and $\mathcal{C}$ be a convergence class on X. For each subset A of X, let $c(A) = \{x \in X \mid (N, x) \in \mathcal{C}, N \text{ in } A\}$. Then c is a closure operator on X. If τ is the topology on X generated by c, then a net N in X converges ($\mathcal{C}$) to x if and only if N converges to x with respect to τ.*

Corollary 1 *Topologies on X and convergence classes on X are in 1–1 correspondence.*

2. Closure operators

One of the primary tools in the proof of Theorem 1 is closure operators. Given a topological space (X, τ), to each subset $A \subseteq X$ we may associate its closure $\overline{A}$, the smallest closed set containing A. Two questions naturally arise: to what extent does $A \longmapsto \overline{A}$ determine the topology τ, and when is a function $c : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ the closure function with respect to some topology on X? (Here $\mathcal{P}(X)$ denotes the set of subsets of X, the power set of X.)

Definition A *closure operator* c on a set X is a function $c : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ with the following properties: for all $A, B \subseteq X$,

a) $c(\emptyset) = \emptyset$,

b) $A \subseteq c(A)$,

c) $c(A) = c(c(A))$,

d) $c(A \cup B) = c(A) \cup c(B)$.

These conditions (*Kuratowski's closure axioms*) are clearly necessary for c to look like the closure function with respect to some topology. They are, in fact, sufficient.

Theorem 2 *Let X be a set and c be a closure operator on X. Let σ be the collection of those $A \subseteq X$ for which $A = c(A)$, and let τ be the collection of complements of elements of σ. Then τ is a topology on X in which $c(A) = \overline{A}$ for all $A \subseteq X$.*

Proof Property a) gives $\emptyset \in \sigma$. Property b) gives $X \in \sigma$. Thus $\emptyset, X \in \tau$. Property d) shows that the union of a finite collection of sets from σ remains in σ. This means that the intersection of a finite collection of sets from τ remains in τ.

Note that if $B \subseteq A$, then $c(B) \subseteq c(A)$. This follows from property d) and the fact that $A = (A \setminus B) \cup B$. Suppose $\{A_i\}_{i \in I}$ is a family of sets from σ and $B = \bigcap_{i \in I} A_i$. For all $i \in I$, $B \subseteq A_i$, hence $c(B) \subseteq \bigcap_{i \in I} c(A_i) = \bigcap_{i \in I} A_i = B$. Property b) gives $B \subseteq c(B)$, and so $B = c(B)$. This shows that the union of a family of sets from τ remains in τ. This concludes the proof that τ is a topology on X.

For $A \subseteq X$, let $\overline{A}$ denote the closure of A with repsect to τ. It remains to show that for all $A \subseteq X$, $c(A) = \overline{A}$. Let $A \subseteq X$. By property c), $c(A) \in \sigma$, whereby $\overline{A} \subseteq c(A)$. Since $\overline{A} \in \sigma$, $c(\overline{A}) = \overline{A}$. Since $A \subseteq \overline{A}$, $c(A) \subseteq c(\overline{A}) = \overline{A}$. Therefore, $c(A) = \overline{A}$. $\square$

Corollary 2 *Topologies on X and closure operators on X are in 1–1 correspondence.*

3. Nets and convergence

The following definitions are meant to be a quick review of directed sets and nets. Please consult [1] for a more comprehensive treatment.

Definition A *directed set* is a non-empty set D with a binary relation $\geq$ which is reflexive, transitive, and which satisfies the following property: for all $d_1, d_2 \in D$, there exists a $d \in D$ such that $d \geq d_1, d_2$.

Definition A *net* in a set X is a pair (N, D) where D is a directed set and $N : D \longrightarrow X$. When the underlying directed set is unimportant, we abbreviate (N, D) by N. In particular, N can stand for both the net and the function on the underlying directed set.

Definition A net (M, E) is a *subnet* of a net (N, D) if there is an $I : E \longrightarrow D$ such that $N \circ I = M$ (the corresponding diagram commutes) and for all $d_0 \in D$, there is an $e_0 \in E$ such that for all $e \geq e_0$, $I(e) \geq d_0$. (The second condition is crafted exactly to make subnets of convergent nets converge.)

Be careful with subnets; they sometimes defy the general intuition and convention of being a sub-object. Still, check that a subnet of a subnet is a subnet. The following is a very natural way to get subnets.

Definition A subset E of a directed set D is *cofinal* if for all $d \in D$, there exists an $e \in E$ such that $e \geq d$. The reader should pause to verify the following facts which we will use later:

- E is a directed set (with the binary relation from D).
- If (N, D) is a net, $(N|_E, E)$ is a subnet of N.
- For all $d_0 \in D$, $\{d \in D \mid d \geq d_0\}$ is cofinal.
- If $D = A \cup B$, then either A or B is cofinal.

For the rest of this section, let X be a topological space and $x \in X$. All nets are in X unless otherwise specified.

Definition A net (N, D) is *eventually in* $A \subseteq X$ if there is a $d_0 \in D$ such that for all $d \geq d_0$, $N(d) \in A$.

Definition A net N *converges to* x if for all neighborhoods U of x, N is eventually in U.

Lemma 1 *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if there is a net in A converging to x.*

The following four lemmas show that conditions i) – iv) for a good notion of convergence $\mathcal{C}$ are necessary if $\mathcal{C}$ is to come from a topology. The proofs of the first three are left as exercises to the reader.

Lemma 2 *Let N be a constant net with value x. Then N converges to x.*

Lemma 3 *Let N be a net and N' be a subnet of N. If N converges to x, then N' converges to x.*

Lemma 4 *Let N be a net and x be a point with the following property: for all subnets N' of N, there exists a subnet N'' of N' such that N'' converges to x. Then N converges to x.*

The next lemma is intimately related to the closure axiom $\overline{A} = \overline{\overline{A}}$, as will be elucidated in the proof of Theorem 1. We require a few more definitions.

Definition The *product directed set* of a family of directed sets $\{(D_\lambda, \geq_\lambda)\}_{\lambda \in \Lambda}$ is the set $\prod_{\lambda \in \Lambda} D_\lambda$ directed by the relation $\geq$ defined by: $p \geq q$ if for all $\lambda \in \Lambda$, $p_\lambda \geq_\lambda q_\lambda$. (The reader must check that the product directed set is actually a directed set with the given relation.)

We will encounter the following product directed set several times. Let D be a directed set, and for each $d \in D$, let E_d be a directed set. Consider the product directed set $D \times \prod_{d \in D} E_d$. We think of elements of this product set as pairs (d', f) where $d' \in D$ and f is a function on D such that for all $d \in D$, f_d is an element of E_d.

Definition Let (N, D) be a net, and for each $d \in D$, let (M_d, E_d) be a net. The *diagonal net of N and the M_d's* is the directed product set $D \times \prod_{d \in D} E_d$ paired with the map $L : D \times \prod_{d \in D} E_d \longrightarrow X$ given by $L(d, f) = M_d(f_d)$. Check that if $A \subseteq X$ is such that for all $d \in D$, (M_d, E_d) is in A, then the diagonal net is in A.

Lemma 5 *Let (N, D) be a net converging to x, and for each $d \in D$, let (M_d, E_d) be a net converging to $N(d)$. Let L be the diagonal net of N and the M_d's. Then L converges to x.*

Proof Let U be an open neighborhood of x. Since N converges to x, there exists a $d_0 \in D$ such that for all $d \geq d_0$, $N(d) \in U$.

For $d \geq d_0$, define $f_d \in E_d$ to be such that for all $e \geq f_d$ in E_d, $M_d(e) \in U$ (this is possible since U is an open neighborhood of $N(d)$ and M_d converges to $N(d)$). For $d \not\geq d_0$, define f_d to be some element of E_d. Now that $f_d \in E_d$ is defined for all $d \in D$, we consider it as an element of $\prod_{d \in D} E_d$.

To prove that L converges to x, we show that for all $(d', f') \geq (d_0, f)$ in $D \times \prod_{d \in D} E_d$, $L(d', f') \in U$. That $(d', f') \geq (d_0, f)$ is that $d' \geq d_0$ and for all $d \in D$, $f'_d \geq f_d$. By definition, $L(d', f') = M_{d'}(f'_{d'})$, which is in U precisely because $d' \geq d_0$ and $f'_{d'} \geq f_{d'}$. □

4. Proof of Theorem 1

First we need a technical lemma.

Lemma 6 *Let X be a set and N, K be two nets in X on the same directed set D. For each $d \in D$, let $D_d = \{d' \in D \mid d' \geq d\}$, X_d be the image of $N|_{D_d}$, and (M_d, E_d) be a net in X_d. Let L be the diagonal net of K and the M_d's. Then L is a subnet of N.*

Proof We must produce a map $I : D \times \prod_{d \in D} E_d \longrightarrow D$ such that $L \circ I = N$ and for all $d_0 \in D$, there is a $(d_1, f_1) \in D \times \prod_{d \in D} E_d$ such that for all $(d, f) \geq (d_1, f_1)$, $I(d, f) \geq d_0$.

Let $(d, f) \in D \times \prod_{d \in D} E_d$. Since L maps into the image of N, the set $N^{-1}[L(d, f)]$ in D is non-empty. Define $I(d, f)$ to be any element of the set $N^{-1}[L(d, f)]$. It follows immediately that $L \circ I = N$.

Let $d_0 \in D$. Fix $f_1 \in \prod_{d \in D} E_d$ and consider $(d_0, f_1) \in D \times \prod_{d \in D} E_d$. Recall that $L(d, f) = M_d(f_d) \in X_d$. If $d \geq d_0$, then $X_d \subseteq X_{d_0}$, and so if $(d, f) \geq (d_0, f_1)$, then $L(d, f) \in X_{d_0}$. By the definitions of X_{d_0} and I, $I(d, f) \geq d_0$. □

Theorem 1 *Let X be a set and $\mathcal{C}$ be a convergence class on X. For each subset A of X, let $c(A) = \{x \in X \mid (N, x) \in \mathcal{C}, N \text{ in } A\}$. Then c is a closure operator on X. If τ is the topology on X generated by c, then a net N in X converges ($\mathcal{C}$) to x if and only if N converges to x with respect to τ.*

Proof We show first that c is a closure operator on X by checking properties a) – d) from section 2.

a) Since directed sets are non-empty, there can be no net in $\emptyset$, and so $c(\emptyset) = \emptyset$. b) Let $A \subseteq X$. A constant net at $x \in A$ is in $\{x\} \subseteq A$ and, by property i), converges ($\mathcal{C}$) to x. Thus $x \in c(A)$, whereby $A \subseteq c(A)$.

c) Let $A \subseteq X$ and $x \in c(c(A))$. Let (N, D) be a net in $c(A)$ which converges ($\mathcal{C}$) to x. For each $d \in D$,

$N(d) \in c(A)$, hence there exists a net (M_d, E_d) in A which converges $(\mathcal{C})$ to $N(d)$. Let L be the diagonal net of N and the M_d's. Then L is in A, and by property iv), L converges $(\mathcal{C})$ to x. This shows $x \in c(A)$, whereby $c(A) = c(c(A))$.

d) Let $A, B \subseteq X$. If $x \in c(A)$, then $x \in c(A \cup B)$ by the definition of c. This shows $c(A) \cup c(B) \subseteq c(A \cup B)$. To see the reverse inclusion, suppose $x \in c(A \cup B)$, and let (N, D) be a net in $A \cup B$ which converges $(\mathcal{C})$ to x. Write $D = D_A \cup D_B$ so that $N(d) \in A$ when $d \in D_A$ and similarly for B. Then either D_A or D_B is cofinal in D, and consequently either $(N|_{D_A}, D_A)$ or $(N|_{D_B}, D_B)$ is a subnet of N which, by property ii), converges $(\mathcal{C})$ to x. Hence either $x \in c(A)$ or $x \in c(B)$, whereby $x \in c(A) \cup c(B)$.

Let τ be the topology on X generated by the closure operator c. We show next that a net N converges $(\mathcal{C})$ to x if and only if N converges to x with respect to the topology τ.

Suppose first that (N, D) converges $(\mathcal{C})$ to x but does not converge to x with respect to τ. Then there is an open neighborhood U of x which N is not eventually in. Hence there is a cofinal subset E of D such that $(N|_E, E)$ is in $X \setminus U$. Since $(N|_E, E)$ is a subnet of N, by property ii), it converges $(\mathcal{C})$ to x. We have exhibited a net in $X \setminus U$ which converges $(\mathcal{C})$ to $x \notin X \setminus U$, hence $X \setminus U \neq c(X \setminus U) = \overline{X \setminus U}$. It follows that $X \setminus U$ is not closed in τ, and hence that U is not open in τ, a contradiction.

Suppose now that a net P converges to x with respect to τ. To show that P converges $(\mathcal{C})$ to x, it suffices by property iii) to show that each subnet of P has a subnet which converges $(\mathcal{C})$ to x.

Let (N, D) be a subnet of P, and let $d \in D$. Let $D_d = \{d' \in D \mid d' \geq d\}$ and X_d be the image of $N|_{D_d}$. Since $(N|_{D_d}, D_d)$ is a subnet of N which is a subnet of P, $N|_{D_d}$ converges to x with respect to τ. Since $(N|_{D_d}, D_d)$ is in X_d, $x \in \overline{X_d} = c(X_d)$. By the definition of c, there exists a net (M_d, E_d) in X_d which converges $(\mathcal{C})$ to x.

We have for each $d \in D$ a net M_d in X_d which converges $(\mathcal{C})$ to x. Let (K, D) be the constant net at x. By property i), K converges $(\mathcal{C})$ to x. Let L be the diagonal net of K and the M_d's. By Lemma 6 and property iv), L is a subnet of N converging $(\mathcal{C})$ to x. $\square$

Corollary 1 *Topologies on X and convergence classes on X are in 1–1 correspondence.*

5. Discussion, examples, and further questions

Theorem 1 tells us that a topology is entirely determined by its collection of convergent nets and their limit points. This justifies the phrase "topology of pointwise convergence" as there is a unique topology in which a net of functions converges if and only if it converges pointwise at each point.

What more, Theorem 1 gives us an inclusion-reversion correspondence between convergence classes and topologies on a set X. Let $\mathcal{C}_1, \mathcal{C}_2$ be convergence classes on X with corresponding topologies τ_1, τ_2. The reader will quickly verify the following facts:

- If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\tau_1 \supseteq \tau_2$.
- The smallest topology containing $\tau_1 \cup \tau_2$ corresponds to the convergence class $\mathcal{C}_1 \cap \mathcal{C}_2$.
- The smallest convergence class containing $\mathcal{C}_1 \cup \mathcal{C}_2$ corresponds to the topology $\tau_1 \cap \tau_2$.

Given a collection $\mathcal{C}$ of nets and points in a set, it may in general be very difficult to decide whether or not $\mathcal{C}$ is a convergence class. In what follows, we provide two important examples of when one can actually verify the conditions. Recall that $\mathcal{C}$ is a convergence class on X if it satisfies the following:

i) If N is a constant net at x, then $(N, x) \in \mathcal{C}$.

ii) If $(N, x) \in \mathcal{C}$, then for all subnets N' of N, $(N', x) \in \mathcal{C}$.

iii) If N is a net and x is a point such that for all subnets N' of N, there is a subnet N'' of N' such that $(N'', x) \in \mathcal{C}$, then $(N, x) \in \mathcal{C}$.

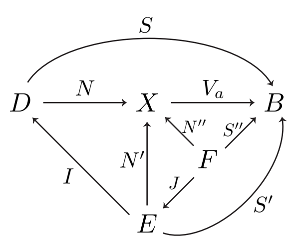
iv) If $((N, D), x) \in \mathcal{C}$ and for each $d \in D$, $(M_d, N(d)) \in \mathcal{C}$, then $(L, x) \in \mathcal{C}$ where L is the diagonal net of N and the M_d's.

Example 1 (One convergence in terms of another) First, let us check that pointwise convergence generates a topology. Let A be a set, B be a topological space, and X be a subset of the set of functions from A to B. For $a \in A$, let $V_a : X \longrightarrow B$ be evaluation at a. Let $\mathcal{C}$ be the collection of pairs (N, f) where N is a net in X, $f \in X$, and for all $a \in A$, $V_a \circ N$ converges (as a net in B) to $f(a)$. The more familiar condition is the special case of sequences: f_n converges ($\mathcal{C}$) to f if and only if f_n converges pointwise to f at each point. Now we check that $\mathcal{C}$ is indeed a convergence class by verifying conditions i) – iv) above.

i) Let be N be a constant net at f, and let $a \in A$. Then $V_a \circ N$ is the constant net at $f(a)$ and hence converges to $f(a)$. Therefore $(N, f) \in \mathcal{C}$.

ii) Let $(N, f) \in \mathcal{C}$, N' be a subnet of N, and $a \in A$. Then it is easily checked that $V_a \circ N'$ is a subnet of $V_a \circ N$, and since $V_a \circ N$ converges to $f(a)$, so does $V_a \circ N'$. Therefore $(N', f) \in \mathcal{C}$.

iii) Let (N, D) be a net and x be a point such that for all subnets N' of N, there is a subnet N'' of N' such that $(N'', x) \in \mathcal{C}$. To show that $(N, x) \in \mathcal{C}$, let $a \in A$ and consider the net $S = V_a \circ N$. It suffices by Lemma 4 to show that for every subnet S' of S, there exists a subnet S'' of S' such that S'' converges to $f(a)$. Let (S', E) be a subnet of S and $I : E \longrightarrow D$ be the map on the index sets. Let $N' : E \longrightarrow X$ be $N' = N \circ I$. Then (N', E) is a subnet of N. Thus there exists a subnet (N'', F) of N' (with index set map $J : F \longrightarrow E$) such that $(N'', f) \in \mathcal{C}$. The net $S'' = V_a \circ N''$ is a subnet of S' and $(N'', f) \in \mathcal{C}$ means that S'' converges to $f(a)$.



iv) Let $((N, D), f) \in \mathcal{C}$ and for each $d \in D$, let $(M_d, N(d)) \in \mathcal{C}$. Let L be the diagonal net of N and the M_d's. Let $a \in A$. Since $V_a \circ N$ converges to $f(a)$ and for all $d \in D$, $V_a \circ M_d$ converges to $N(d)(a) = (V_a \circ N)(d)$, it suffices by Lemma 5 to show that $V_a \circ L$ is the diagonal net of $V_a \circ N$ and the $V_a \circ M_d$'s. Let P be the diagonal net of $V_a \circ N$ and the $V_a \circ M_d$'s. Both L and P are functions on the same directed set $D \times \prod_{d \in D} E_d$. Moreover, $P : D \times \prod_{d \in D} E_d \longrightarrow B$ is defined by $P(d, g) = (V_a \circ M_d)(g_d) = V_a(M_d(g_d)) = V_a(L(d, g)) = (V_a \circ L)(d, g)$. Hence $P = (V_a \circ L)$.

Alternatively, to see that $\mathcal{C}$ is a convergence class, one could show that it consists of exactly the pairs of convergent nets and their limits points in the topology which X inherits from the product topology on B^A. This is arguably an easier approach, but because the situation can be generalized considerably, it is actually the mechanics of this exercise that are important and interesting. Let's see another example.

Let $K \subseteq \mathbb{R}^n$ be compact, and consider $C_c^\infty(K)$, the space of all smooth functions with compact support in K. There is a natural topology on this space generated by the family of semi-norms indexed by the multi-index α: $||\varphi||_\alpha = ||\partial^\alpha \varphi||_u$. It is an exercise then to show that a net N in $C_c^\infty(K)$ converges to f

in this topology if and only if $\partial^\alpha \circ N$ converges to $\partial^\alpha f$ uniformily for all multi-indices α.

Let $\mathcal{D}$ be the collection of pairs (N, f) where N is a net in $C_c^\infty(K)$, $f \in C_c^\infty(K)$, and for all multi-indices α, $\partial^\alpha \circ N$ converges (as a net in $C_c^\infty(K)$) to $\partial^\alpha f$ in the uniform topology on $C_c^\infty(K)$. The proof that $\mathcal{D}$ is a convergence class follows the exact line of reasoning above, and so we know immediately that $\mathcal{D}$ generates a topology on $C_c^\infty(K)$.

More generally, let X be a set, B be a topological space, and $\{M_i : X \longrightarrow B\}_{i \in I}$ be a family of maps. Let $\mathcal{C}$ be the collection of pairs (N, x) where N is a net in X, $x \in X$, and for all $i \in I$, $M_i \circ N$ converges (as a net in B) to $M_i(x)$. Then by the same reasoning as above, $\mathcal{C}$ is a convergence class. Many examples of convergence, especially from functional analysis, fit this form. These include: pointwise, semi-norm, weak, weak-*, weak operator, and strong operator convergences. □

Example 2 (There is no topology of pointwise, almost everywhere convergence) Let X be the space of measurable functions from $[0,1]$ to $\mathbb{C}$. Let $\mathcal{C}$ be the collection of pairs (N, f) where N is a net in X, $f \in X$, and there exists a subset $H \subseteq [0,1]$, depending on N, of measure 0 such that for all $a \in [0,1] \setminus H$, $V_a \circ N$ converges (as a net in $\mathbb{C}$) to $f(a)$.

We want to show that $\mathcal{C}$ is not a convergence class by showing that $\mathcal{C}$ does not meet condition iv). Let $D = (0,1)^2$ ordered lexicographically, and let (N, D) be the constant net at 0, the zero function. Clearly N converges ($\mathcal{C}$) to 0. For each $d = (d_1, d_2) \in D$, let $M_d : \mathbb{N} \longrightarrow X$ be the constant net at δ_{d_2}, the point mass at d_2. Clearly M_d converges ($\mathcal{C}$) to 0. Let L be the diagonal net of N and the M_d's.

Suppose for a contradiction that L converges ($\mathcal{C}$) to 0. There exists an $H \subseteq [0,1]$ of measure 0 such that for all $a \in [0,1] \setminus A$, $V_a \circ L$ converges to 0. Fix $a \in (0,1) \setminus H$, and let $0 < \epsilon < 1$. There exists a $(d', g') \in D \times \prod_{d \in D} \mathbb{N}$, such that for all $(d, g) \geq (d', g')$, $|V_a(L(d, g))| < \epsilon$. Let $d \in D$ where $d_1 > d_1'$ and $d_2 = a$. Then $d > d'$, so $(d, g) \geq (d', g)$. Therefore $|V_a(L(d, g))| < \epsilon$. But $V_a(L(d, g)) = \delta_a(a) = 1$, a contradiction. □

We are left with some natural open ends:

- Let $\mathcal{D}$ be an arbitrary collection of nets and points. Call $\mathcal{D}$ *consistent* if there is a convergence class $\mathcal{C}$ containing it. Are there conditions on a collection of nets and points to ensure that it is consistent? Since the intersection of convergence classes is a convergence class, there is a smallest convergence class containing $\mathcal{D}$ if it is consistent; call it the convergence class *generated* by $\mathcal{D}$. Is there anything useful in this direction?
- Are there natural further conditions that can be placed on a convergence class $\mathcal{C}$ which will guarantee that the generated topology will be first countable, metrizable, etc...?
- Is there a well-known topology describable only in terms of its notion of convergence? It would make a great example to cite in these notes.

References

[1] John L. Kelley, *General topology*, Springer-Verlag, New York, 1975. MR 0370454 (51 #6681)