Sets of Recurrence

Daniel Glasscock, November 2012

Theorem (Poincaré) *Let $(X, \mathcal{M}, \mu, T)$ be a finite measure preserving system, and let $A \in \mathcal{M}$ with $\mu(A) > 0$. There exists an $n \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-n}A) > 0.$$

Proof The sets A, $T^{-1}A$, $T^{-2}A$, … all have measure $\mu(A) > 0$ in a space X of finite measure. Therefore, there exists $i < j \in \mathbb{N}$ such that $\mu\left(T^{-i}A \cap T^{-j}A\right) > 0$. Now

$$\mu\left(T^{-i}A \cap T^{-j}A\right) = \mu\left(T^{-i}(A \cap T^{-(j-i)}A)\right) = \mu\left(A \cap T^{-(j-i)}A\right) > 0.$$

Definition A set $R \subset \mathbb{N}$ is a *set of recurrence* if for all finite measure preserving systems $(X, \mathcal{M}, \mu, T)$ and for all $A \in \mathcal{M}$ with $\mu(A) > 0$, there exists an $r \in R$ such that $\mu(A \cap T^{-r}A) > 0$.

So, Poincaré's theorem is exactly that $\mathbb{N}$ is a set of recurrence. An examination of the proof leads one to see that if $E \subseteq \mathbb{N}$ is infinite, then the set of positive differences $\{e_2 - e_1 \mid e_1 < e_2 \in E\}$ is a set of recurrence. In fact, R is a set of recurrence if it contains difference sets of arbitrarily large sets of positive integers. It is also readily seen that for all $a \in \mathbb{N}$, $a\mathbb{N}$ is a set of recurrence. What more, any set of recurrence intersects all $a\mathbb{N}$ non-trivially (consider permutations of a finite set).

The three theorems that follow give non-trivial examples of sets of recurrence.

Theorem (Furstenberg) *If $p(x) \in \mathbb{N}[x]$ is a polynomial with zero constant term, then $p(\mathbb{N}) = \{p(n) \mid n \in \mathbb{N}\}$ is a set of recurrence.*[1]

Theorem (Kamae-Mendes France) *Let $\mathbb{P}$ be the set of primes. Then $\mathbb{P} \pm 1$ are both sets of recurrence.*[2]

Theorem *If α is irrational, then $\{\lfloor n^2\alpha \rfloor \mid n \in \mathbb{N}\}$ is a set of recurrence.*[3]

For many more examples of sets of recurrence, see [3].

We will now explore an interesting application of sets of recurrence to number theory. The *upper density* of a set $E \subseteq \mathbb{N}$ is

$$\overline{d}(E) = \limsup_{N \to \infty} \frac{|E \cap [1, N]|}{N}.$$

Consider the system $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \overline{d}, T)$ where $T(n) = n + 1$. Though it is not a measure preserving system ($\overline{d}$ is not countably additive), it exhibits some of the same recurrence properties as one. For example, the following is an exercise: given $E \subseteq \mathbb{N}$ with $\overline{d}(E) > 0$, there exists an $n \in N$ such that $\overline{d}\left(E \cap (E - n)\right) > 0$.

The connection between recurrence in measure preserving systems and in $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \overline{d}, T)$ is made much more precise by the following.

Theorem (Furstenberg's Correspondence Principle) *Let $E \subseteq \mathbb{N}$ with $\overline{d}(E) > 0$. There exists a finite measure preserving system $(X, \mathcal{M}, \mu, T)$ and a set $A \in \mathcal{M}$ with $\mu(A) = \overline{d}(E)$ such that for all $n \in \mathbb{N}$,*

$$\mu\left(A \cap T^{-n}A\right) \leq \overline{d}\left(E \cap (E - n)\right).$$

[1] Furstenberg's proof (see [4]) utilizes the spectral theorem for unitary operators on Hilbert spaces. (To see the connection, note that $\mu(A \cap T^{-n}A) = \langle U^n \chi_A, \chi_A \rangle$ where $Uf = f \circ T$ is a unitary operator on $L^2(X, \mu)$.) Bergelson (see [1]) gives an alternative proof utilizing a splitting of $L^2(X, \mu)$ in the spirit of von Neumann's proof of the mean ergodic theorem.

[2] Kamae and Mendes France in [5] show something stronger: $\mathbb{P} \pm 1$ are *van der Corput* sets. The fact that van der Corput sets are sets of recurrence was "known to many in Jerusalem" around this time, according to Bergelson. Note that $\mathbb{P}$ is not a set of recurrence since it misses $4\mathbb{N}$. In fact, it is known that $\mathbb{P} \pm a$ is not a set of recurrence for any $a \neq \pm 1$.

[3] Images of a wide range of *generalized polynomials* (polynomials involving floor functions) are sets of recurrence. See [2], for example.

Corollary *Let $E \subseteq \mathbb{N}$ with $\overline{d}(E) > 0$, and let R be a set of recurrence. Then $R \cap (E - E) \neq \emptyset$.*[4]

Proof: By Furstenberg's correspondence principle and the fact that R is a set of recurrence, there is an $r \in R$ such that $\overline{d}\,(E \cap (E - r)) > 0$. In particular, $E \cap (E - r) \neq \emptyset$. But $E \cap (E - r) \neq \emptyset$ if any only if $r \in E - E$.

Let $E \subseteq \mathbb{N}$ with $\overline{d}(E) > 0$. By the corollary, if R is a set of recurrence, we know that some element of R will appear as a difference between elements of E. Each time a new set of recurrence is identified, we learn more about differences in sets of integers with positive upper density!

Applying the first of the three theorems above, we have the following.

Theorem (Furstenberg) Let $p(x) \in \mathbb{N}[x]$ be a polynomial with zero constant term. If $E \subseteq \mathbb{N}$ with $\overline{d}(E) > 0$, then there is an $n \in N$ such that $p(n) \in E - E$.[5]

The other two theorems above apply similarly: in a set $E \subseteq \mathbb{N}$ with $\overline{d}(E) > 0$, we are guaranteed to find differences of the form $p - 1$ and $\lfloor n^2 \alpha \rfloor$ for some $p \in \mathbb{P}$, $n \in \mathbb{N}$.

The most famous application of this connection is Furstenberg's proof of Szemerédi's theorem.

Theorem (Szemerédi) *If $E \subseteq \mathbb{N}$ with $\overline{d}(E) > 0$, then E contains arbitrarily long arithmetic progressions.*

Furstenberg's proof makes use of a (slightly) more general correspondence principle coupled with the following much harder multiple recurrence theorem.

Theorem (Furstenberg) *Let $(X, \mathcal{M}, \mu, T)$ be a finite measure preserving system, and let $A \in \mathcal{M}$ with $\mu(A) > 0$. For all $k \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-(k-1)n}A) > 0.$$

References

[1] Vitaly Bergelson, *Ergodic Ramsey theory—an update*, Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994), London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996, pp. 1–61. MR 1411215 (98g:28017)

[2] Vitaly Bergelson and Inger Johanne Håland, *Sets of recurrence and generalized polynomials*, Convergence in ergodic theory and probability (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., vol. 5, de Gruyter, Berlin, 1996, pp. 91–110. MR 1412598 (98g:28026)

[3] Vitaly Bergelson and Emmanuel Lesigne, *Van der Corput sets in $\mathbb{Z}^d$*, Colloq. Math. **110** (2008), no. 1, 1–49. MR 2353898 (2008j:11089)

[4] Harry Furstenberg, *Poincaré recurrence and number theory*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), no. 3, 211–234. MR 628658 (83d:10067)

[5] T. Kamae and M. Mendès France, *van der Corput's difference theorem*, Israel J. Math. **31** (1978), no. 3-4, 335–342. MR 516154 (80a:10070)

[4]The converse is true: if $R \subseteq \mathbb{N}$ is such that for all $E \subseteq \mathbb{N}$ with $\overline{d}(E) > 0$, $R \cap (E - E) \neq \emptyset$, then R is a set of recurrence.

[5]The special case $p(n) = n^2$ was proven independently by Sárközy with harder analytic techniques.