

The Problem of Finite Invariant Measures

Daniel Glasscock, February 2013

A central topic in ergodic theory is understanding measure preserving transformations of finite measure spaces. In some cases, the body of results developed to this end may apply to transformations which are not measure preserving. The existence of a finite invariant measure is one of those cases and is discussed below. The primary reference is [3].

Let (X, \mathcal{M}, μ) be a finite measure space and $T : X \rightarrow X$ be measurable (but not necessarily measure preserving). Suppose there exists a finite measure λ on \mathcal{M} which is T -invariant (that is, $\lambda(T^{-1}E) = \lambda(E)$) and which is equivalent to μ (write $\lambda \sim \mu$ meaning $\lambda(E) = 0$ if and only if $\mu(E) = 0$). It is an exercise to verify the following hold:

- (Recurrence) If $\mu(E) > 0$, then there exists an $n \in \mathbb{N}$ such that $\mu(E \cap T^{-n}E) > 0$. All similar results on multiple recurrence hold as well.
- (Pointwise Ergodic Theorem) For all $f \in L^\infty(\mu)$, there exists a T -invariant $f^* \in L^1(\mu)$ such that for μ -a.e. $x \in X$, $\frac{1}{N} \sum_1^N f(T^i x) \rightarrow f^*(x)$. This of course holds for all $f \in L^1(\lambda)$, but knowing this is only helpful when we have some handle on the invariant measure λ .
- (Mean Ergodic Theorem) For all $f \in L^\infty(\mu)$, $\frac{1}{N} \sum_1^N U_T f \rightarrow P_T f$ in $L^2(\lambda)$, where P_T is the orthogonal projection onto $\{f \in L^2(\lambda) \mid f \circ T = f\}$. Here again this holds for $f \in L^2(\lambda)$. It seems that some handle on λ is necessary to make use of this result.

Much is gained by having the existence (and especially a description) of an equivalent, finite invariant measure. The question is: when do such measures exist? For simplicity, we will assume that T is invertible. Also, without loss of generality (see [5]), we may assume that T is non-singular; that is, if $\mu(E) = 0$, then $\mu(T E) = \mu(T^{-1} E) = 0$.

Given a finite measure space (X, \mathcal{M}, μ) and an invertible, non-singular, measurable transformation $T : X \rightarrow X$, when does there exist a finite measure λ on (X, \mathcal{M}) which is T -invariant and equivalent to μ ?

A complete solution to this problem is a characterization of those systems (X, \mathcal{M}, μ, T) which admit such a λ . To keep from repeating, make the following abbreviation.

Inv: There exists a finite measure λ on (X, \mathcal{M}) which is T -invariant and equivalent to μ .

In what follows, we outline 4 different complete solutions.

1. The first solution was given by Eberhard Hopf in 1932. We introduce an equivalence relation on \mathcal{M} by writing $E \sim F$ if and only if there exist countable partitions $E = \cup_i E_i$, $F = \cup_i F_i$, and a sequence of integers $(n_i)_i$ such that $T^{n_i} E_i = F_i$. In this case, we say that E and F are *equivalent under countable decomposition*. A set is *incompressible* if it is not equivalent to a proper subset of itself; in other words, E is incompressible if $E \sim F$ and $F \subseteq E$ imply that $\mu(E \setminus F) = 0$.

Theorem 1 (Hopf [6]): **Inv** if and only if X is incompressible.

The incompressibility of X is clearly necessary for the existence of an invariant measure; the reader is referred to Hopf's original argument for sufficiency. This solution paints a nice geometric picture: if T does not "shrink" sets with respect to μ (in this countable partition sense), then there is a way to measure sets which is invariant with respect to T . The problem here is that there are practically no tools for proving the incompressibility of a space.

2. As noted above, if **Inv**, then a version of Birkhoff's pointwise ergodic theorem holds for μ and T . Thus a necessary condition for **Inv** is that for all $E \in \mathcal{M}$, $\frac{1}{N} \sum_1^N \chi_E \circ T^i$ converges μ -a.e.. It turns out

that this condition is also sufficient.

Theorem 2 (Dowker [1]): **Inv** if and only if for all $E \in \mathcal{M}$, $\frac{1}{N} \sum_1^N \chi_E \circ T^i$ converges μ -a.e..

Proof (sketch). Suppose that for every $E \in \mathcal{M}$, $\frac{1}{N} \sum_1^N \chi_E \circ T^i$ converges μ -a.e., and let $f_E \in L^1(\mu)$ be its limit. Define

$$\lambda(E) = \int_X f_E d\mu.$$

(This definition is inspired from the necessary condition; if **Inv**, then $f_E = \mathbb{E}_\lambda(\chi_E | \mathcal{F})$, where \mathcal{F} is the σ -algebra generated by T -invariant sets, and so $\lambda(E) = \int_X f_E d\lambda$.) We then show that λ is a finite, T -invariant measure (making use of the Vitali-Hahn-Saks Theorem) which is equivalent to μ . The reader is referred to [3] for the details. \square

This solution is also difficult to apply because it requires one to check pointwise almost everywhere convergence for a large class of functions. It does, however, lead to the understanding of an invariant measure: the averages $\frac{1}{N} \sum_1^N \chi_E \circ T^i$ measure the distribution of orbits of T , and if they behave nicely, we may use them to measure sets in a T -invariant way.

3. Instead of considering averages along orbits of points, we may consider recurrence in orbits of sets. Poincaré recurrence (as well as multiple recurrence) holds in systems for which **Inv**; we are led to ask whether these recurrence theorems are sufficient for **Inv**.

Call T *conservative* if for all $E \in \mathcal{M}$ with $\mu(E) > 0$, there exists an $n \in \mathbb{Z}$ such that $\mu(E \cap T^n E) > 0$; in other words, conservative transformations are exactly those for which Poincaré recurrence holds. The question now is whether **Inv** is equivalent to a system being conservative. The answer is no, as illustrated by the following example.

Claim: There exist conservative systems in which **Inv** does not hold.

Proof (Halmos [5]). Consider a measure space $(\mathbb{R}, \mathcal{B}, \eta)$ where \mathcal{B} is the σ -algebra of Borel subsets of \mathbb{R} and η is a finite measure equivalent to the standard Lebesgue measure m . (Any η will do; take, for example, $\eta(E) = \int_E \frac{1}{x^2+1} dm(x)$.) Let T be an invertible, ergodic (this means: $m(E \cap B^{-1}E) = 0$ implies $m(E) = 0$ or $m(\mathbb{R} \setminus E) = 0$), measure preserving transformation of (X, \mathcal{B}, m) . We claim that T on $(\mathbb{R}, \mathcal{B}, \eta)$ is conservative but that **Inv** does not hold.

First we show that T is conservative. Note that T is η -ergodic since it is m -ergodic and $\eta \sim m$. If $(\mathbb{R}, \mathcal{B}, \eta, T)$ is not conservative, then there exists a set $E \in \mathcal{B}$ with $\eta(E) > 0$ such that the translates of E , $T^n E$ for $n \in \mathbb{Z}$, are all disjoint. Since η is non-atomic, there exists an $F \subseteq E$ with $0 < \eta(F) < \eta(E)$. Then $\cup_n T^n F$ is a non-trivial, T -invariant set, a contradiction with the ergodicity of T .

Next we show that **Inv** does not hold. Suppose that λ is a finite, T -invariant measure equivalent to η . Since $\eta \sim m$, $\lambda \sim m$. By Radon-Nikodym, there exists an $f \in L^1(m)$ such that for all $E \in \mathcal{B}$, $\lambda(E) = \int_E f dm$. Since λ is T -invariant,

$$\int_E f \circ T dm = \int_{TE} f dm = \lambda(TE) = \lambda(E) = \int_E f dm.$$

This holds for all $E \in \mathcal{B}$, therefore f is T -invariant. Since T is ergodic, f is a.e. constant; since $f \in L^1(m)$, $f = 0$. This implies $\lambda = 0$, a contradiction with $\lambda \sim m$. \square

A slightly stronger recurrence condition, however, does suffice for **Inv**. A set which does not recur at all is called *wandering*. Generalizing, a set with infinitely many disjoint translates is called *weakly wandering*; that is, $E \in \mathcal{M}$ is weakly wandering if there exists an infinite sequence of integers $(n_i)_i$ such that when $i \neq j$, $\mu(T^{n_i} E \cap T^{n_j} E) = 0$. By the pigeon-hole principle, weakly wandering sets of positive

measure do not exist in finite measures spaces for which **Inv**. It is a beautiful fact that the converse is also true.

Theorem 3 (Hajian-Kakutani [4]): **Inv** if and only if there are no weakly wandering sets of positive measure.

The condition that there are no weakly wandering sets of positive measure is exactly that for all $E \in \mathcal{M}$ with $\mu(E) > 0$ and all $(n_i)_i \subseteq \mathbb{Z}$, there exist $i \neq j$ such that $\mu(T^{n_i}E \cap T^{n_j}E) > 0$. Hajian and Kakutani's result says that if large sets remain large under "enough" translates of T , then there exists an equivalent, finite invariant measure. The proof of Theorem 3 will follow immediately from the following lemma combined with a result of Dowker discussed below.

Lemma (Hajian-Kakutani [4]) Suppose $\mu(E) > 0$ and $\liminf_n \mu(T^n E) = 0$. Then for all $\epsilon > 0$, there exists an $F_\epsilon \subseteq E$ with $\mu(F_\epsilon) < \epsilon$ such that $E \setminus F_\epsilon$ is weakly wandering.

Proof of Lemma (sketch). Note that because T is non-singular, for all $n \in \mathbb{Z}$, $T^n \mu \sim \mu$. We create an increasing sequence $(n_i)_{i \geq 0} \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$ and all $0 \leq j < i$, $\mu(T^{n_i - n_j} E) < \frac{\epsilon}{i^2}$. Let

$$F_\epsilon = E \cap \left(\bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} T^{n_i - n_j} E \right).$$

It is straightforward to check that $\mu(F_\epsilon) < \epsilon$ and that $E \setminus F_\epsilon$ is weakly wandering along $(n_i)_i$. □

4. The expression $\liminf_n \mu(T^n E)$ gives a lower bound on the eventual size of the translates of E and quantifies the "enough" appearing above. If **Inv** and $\mu(E) > 0$, then it is an exercise to show that $\liminf_n \mu(T^n E) > 0$; this condition also suffices for **Inv**.

Theorem 4 (Dowker [2]): **Inv** if and only if for all $E \in \mathcal{M}$ with $\mu(E) > 0$, $\liminf_n \mu(T^n E) > 0$.

The proof utilizes Banach limits and a result for extending a finitely additive measure to a σ -additive one. The reader is referred to [3] for the details. This result was strengthened by Sucheston in [7]. He showed that it suffices for **Inv** that for all $E \in \mathcal{M}$ with $\mu(E) > 0$, $\limsup_{N-M \rightarrow \infty} \frac{1}{N-M} \sum_M^{N-1} \mu(T^i E) > 0$. The reader is referred to Sucheston's paper for the details.

References

- [1] Yael Naim Dowker. Finite and σ -finite invariant measures. *Ann. of Math. (2)*, 54:595–608, 1951.
- [2] Yael Naim Dowker. On measurable transformations in finite measure spaces. *Ann. of Math. (2)*, 62:504–516, 1955.
- [3] Nathaniel A. Friedman. *Introduction to ergodic theory*. Van Nostrand Reinhold Co., New York, 1970. Van Nostrand Reinhold Mathematical Studies, No. 29.
- [4] Arshag B. Hajian and Shizuo Kakutani. Weakly wandering sets and invariant measures. *Trans. Amer. Math. Soc.*, 110:136–151, 1964.
- [5] Paul R. Halmos. *Lectures on ergodic theory*. Chelsea Publishing Co., New York, 1960.
- [6] Eberhard Hopf. Theory of measure and invariant integrals. *Trans. Amer. Math. Soc.*, 34(2):373–393, 1932.
- [7] Louis Sucheston. On existence of finite invariant measures. *Math. Z.*, 86:327–336, 1964.