Exponentiales Replicatas

Ref Euler, De formulis exponentialibus replicatis (on unraveling exponential formulas), Acta Academiae Scientarum Imperialis Petropolitinae 1, 1778, pp 38-60 (E489)

My translation of §1 (very rough and likely wrong)

The Academy recently communicated with the famous Marquis de Condorcet (Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet, 1743-1794 French philosopher, politician, mathematician) regarding the analytic formulas presented below which may be called repeated exponentiation. Since every power goes into the exponent following it, the expression is usually written $r^{r^{r^{r}}}$. Despite the incredible force of the investigation, still not enough is known about the nature of such expressions. This account is useful in that it explains some special properties of such expressions.

Euler's "repeated exponentials" is today known as "tetration".
Tetration is the 4th "hyper-operator".

Hyper operator order	Name	Notation	Definition	
0	successor	$a \circ n$	$n+1$ $\cancel{a+1}$ "next integer" function.	
1	addition	$a+n$	$\underbrace{a \circ a \circ \cdots \circ a}_{n+1 \text{ times}}$	
2	multiplication	$a \times n$	$\underbrace{a + a + \cdots + a}_{n \text{ times}}$	
3	exponentiation	a^n $(a \uparrow n)$ †	$\underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$	non-commutative!
4	tetration ✱	$a \uparrow\uparrow n$	$\underbrace{a \uparrow (a \uparrow \cdots ((\uparrow a)))}_{n \text{ times}}$	non-associative! non-commutative!
5	pentation	$a \uparrow\uparrow\uparrow n$	$\underbrace{a \uparrow\uparrow a \uparrow\uparrow \cdots \uparrow\uparrow a}_{n \text{ times}}$	non-associative! non-commutative!
⋮	⋮	⋮		

$$\boxed{a^{a^{a^{a^{\cdot^{\cdot^{a}}}}}} = a \uparrow a \uparrow \cdots \uparrow a}$$

✱ Nomenclature due to Reuben Goodstein

† Notation due to Donald Knuth

Before we continue, it is good to recall how a^α is defined for positive real a, α.

Two ways: • a^n defined in terms of multiplication for $n=1,2,\dots$

$a^{\frac{1}{n}}$ is the unique positive real solution to $x^n = a$

$a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n$ by rules of exponentiation.

$a^\alpha = \lim_{r\to\alpha} a^r$ where $r \in \mathbb{Q}$. In other words,

a^α defined by extending exponentiation by rationals continuously to the reals.

• Alternatively, ~~knowing~~ defining $\log a = \int_1^a t^{-1}\,dt$, we can define $a^\alpha = e^{\alpha \log a}$.

Keep in mind that a^{a^a} written without parentheses means $a^{(a^a)}$ so that the exponentials are evaluated from "top to bottom", so to speak.

Euler, motivated by Condorcet, considered the first question that comes to mind when writing an iterated exponential:

$$\boxed{\text{Q: For which } a>0 \text{ does } a^{a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}} = \lim_{n\to\infty} a\uparrow\uparrow n \text{ converge?}}$$

To expose the subtlety here, consider the ~~prob~~ following problem which appeared as a high school math contest problem:

For which real numbers a does $a^{a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}} = 8$?

It looks like one could solve this as

$$a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = 8 \Rightarrow a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = a^{\left(a^{a^{\cdot^{\cdot^{\cdot}}}}\right)} = a^8 = 8$$

and so if $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$ converges to 8, then $a = 8^{\frac{1}{8}}$.

How do we know $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = \lim_{n\to\infty} a\uparrow\uparrow n$ converges for $a = 8^{\frac{1}{8}}$?

It, in fact, does converge for $a = 8^{\frac{1}{8}}$, but the limit is not 8!

Let's consider this a bit more carefully:

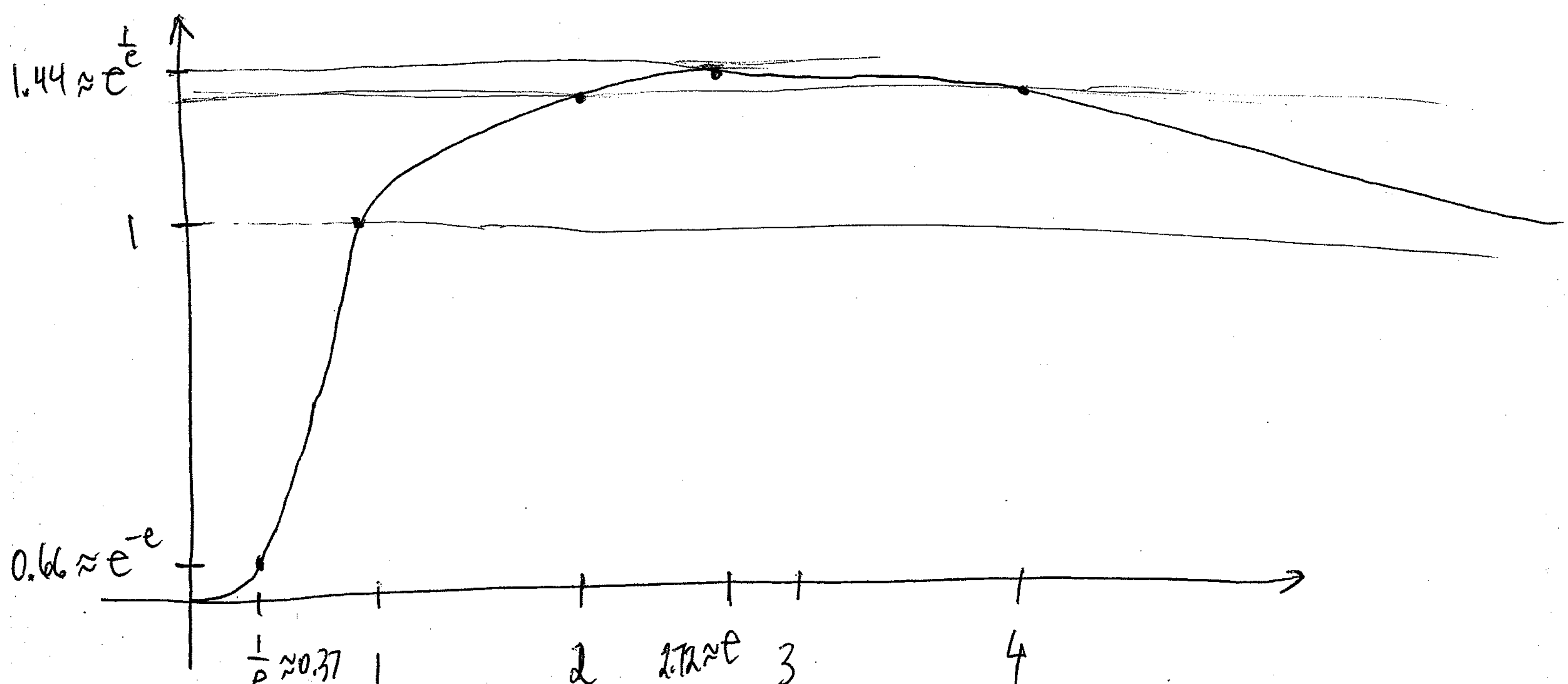
If $\lim_{n\to\infty} a\uparrow\uparrow n$ exists and is equal to b, then

$$b = \lim_{n\to\infty} a\uparrow\uparrow(n+1) = \lim_{n\to\infty} a^{a\uparrow\uparrow n} = a^{\lim_{n\to\infty} a\uparrow\uparrow n} = a^b,$$

so $a^b = b$ or $a = b^{\frac{1}{b}}$.

$$\boxed{\text{In other words, } x \longmapsto \lim_{n\to\infty} x\uparrow\uparrow n \text{ is the inverse of the function } f(x) = x^{\frac{1}{x}} \text{ on its domain.}}$$

Consider the graph of $f(x) = x^{\frac{1}{x}}$.



It's worth taking a tangent at this point to talk about this ~~graphs~~ function's connection to another interesting equation:

$$\boxed{x^y = y^x} \longleftrightarrow \boxed{x^{\frac{1}{x}} = y^{\frac{1}{y}}}$$

This equation appears in a letter from Daniel Bernoulli to Christian Goldbach in 1728. In it, Bernoulli explains (states) that there is only 1 integer solution $2^4 = 4^2$ but many rational solutions.

Goldbach responded with a parameterization of all solutions obtained in the following way: if $y > x$, then $y = sx$ for some $s > 1$. then $x^y = y^x$ becomes $x^{sx} = (sx)^x$, which in turn is $x^s = sx$, or $\begin{cases} x = s^{\frac{1}{s-1}} \\ y = s^{\frac{s}{s-1}} \end{cases}$ $s > 1$.

Euler took this a step further and parameterized a family of rational solutions by putting $s = 1 + \frac{1}{n}$ to get $\begin{cases} x = \left(1 + \frac{1}{n}\right)^n \\ y = \left(1 + \frac{1}{n}\right)^{n+1} \end{cases}$.

[S. Hurwitz showed that these are all of the rational solutions to $x^y = y^x$ in 1967!!]

Euler also remarked on the connection between the solutions of $x^y \overset{(*)}{=} y^x$ and the related equation $x^{x} \overset{(**)}{=} y^y$

In particular, he argues that (a,b) is a solution to $(*)$ if and only if $(\frac{1}{a}, \frac{1}{b})$ is a solution to $(**)$. Indeed,

$$a^b = b^a \Leftrightarrow \left(\frac{1}{\frac{1}{a}}\right)^{\frac{1}{\frac{1}{b}}} = \left(\frac{1}{\frac{1}{b}}\right)^{\frac{1}{\frac{1}{a}}} \Leftrightarrow \left(\frac{1}{b}\right)^{\frac{1}{b}} = \left(\frac{1}{a}\right)^{\frac{1}{a}}.$$

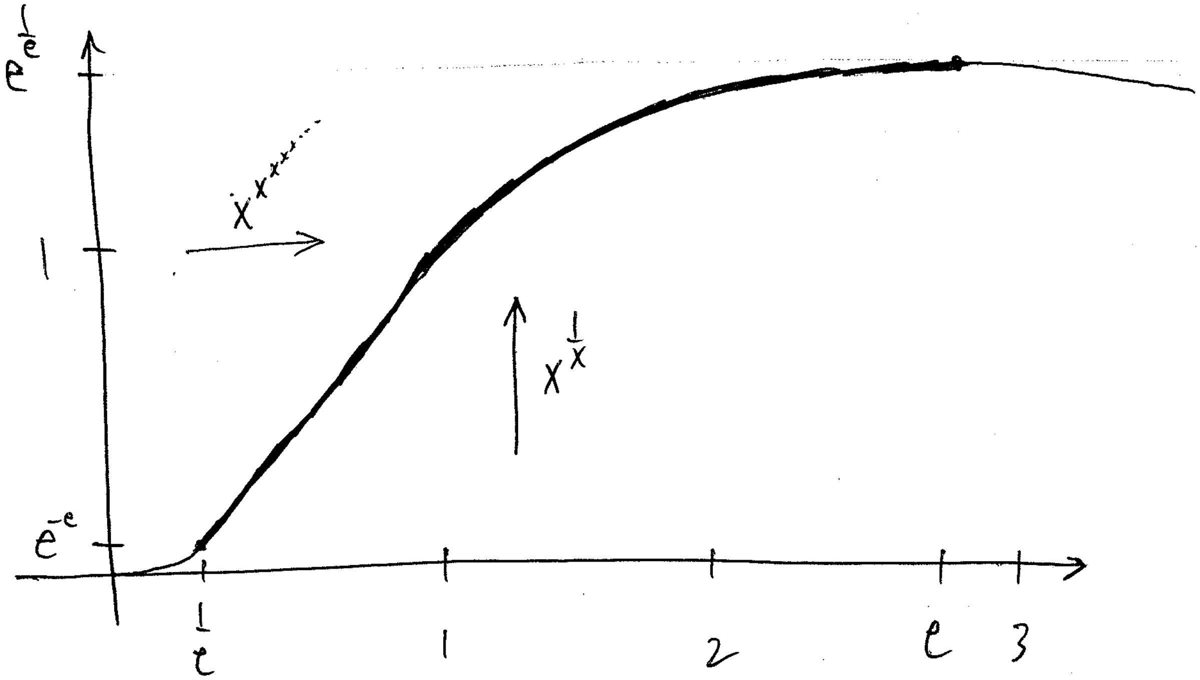
In particular, the solutions to $x^y = y^x$ and $x^x = y^y$ are in 1-1 correspondence!

Let's return to the convergence of $a^{a^{a^{a^{\cdots}}}}$

Since $x \mapsto \lim_{n\to\infty} x \uparrow\uparrow n$ is the inverse of $f(x) = x^{\frac{1}{x}}$ on its domain and the range of $f(x)$ is contained in $[0, e^{\frac{1}{e}}]$, <u>the domain of $x \mapsto \lim_{n\to\infty} x\uparrow\uparrow n$ is contained in $[0, e^{\frac{1}{e}}]$.</u>

The following theorem was ~~known~~ argued first by Euler:

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\quad a^{a^{a^{a^{\cdots}}}} \text{ converges if and only if } e^{-e} \leq a \leq e^{\frac{1}{e}}. \\ \text{Moreover, if } a^{a^{a^{a^{\cdots}}}} = b, \text{ then } \frac{1}{e} \leq b \leq e. \text{ and } b^{\frac{1}{b}} = a.\end{array}}$$



The proof of the theorem is broken into two parts depending on whether $a>1$ or $a<1$.

proof (in the case that $a>1$):

Since we already know $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$ diverges if $a>e^{\frac{1}{e}}$, let $1<a\leq e^{\frac{1}{e}}$.

Suppose we know $a, a^a, a^{a^a}, \ldots$ is increasing and bounded by e.

Then, since a bounded increasing sequence converges, $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}=b$ where $b^{\frac{1}{b}}=a$ and $b\leq e$, which is what we wanted to show.

The sequence is increasing because:

$$1<a \Rightarrow a^1<a^a \Rightarrow a^{a^1}<a^{a^a} \Rightarrow \cdots$$

To see that it is bounded, note that $a\leq e^{\frac{1}{e}}<e$, so

$$a^a \leq \left(e^{\frac{1}{e}}\right)^a < \left(e^{\frac{1}{e}}\right)^e = e$$

$$a^{a^a} \leq \left(e^{\frac{1}{e}}\right)^{a^a} < \left(e^{\frac{1}{e}}\right)^e = e$$

$$\vdots$$

This shows $a, a^a, a^{a^a}, \ldots$ is both increasing and bounded. ∎

The case when $a<1$ is more difficult.

If $0<a<1$, then $a^1 < a^a < a^0$, meaning $0<a<a^a<1$. This implies $a^1 < a^{a^a} < a^a < a^0$, so $0<a<a^{a^a}<a^a<1$. Again, $a^1 < a^{a^a} < a^{a^{a^a}} < a^a < a^0$, so

$$0<a<a^{a^a}<a^{a^{a^a}}<a^a<1.$$

In general, $0<a<a^{a^a}<\cdots<a\uparrow\uparrow(2n-1)<a\uparrow\uparrow(2n)<\cdots<a^a<1$.

In other words, the sequence of odd terms $\{a\uparrow\uparrow(2n-1)\}$ is increasing and the sequence of even terms $\{a\uparrow\uparrow(2n)\}$ is decreasing. Since both sequences are bounded, they both converge.

Let $b_{odd} = \lim_{n\to\infty} a\uparrow\uparrow(2n-1)$, $b_{even} = \lim_{n\to\infty} a\uparrow\uparrow(2n)$.

Then $b_{odd} \leq b_{even}$ and $\lim_{n\to\infty} a\uparrow\uparrow n$ exists if and only if $b_{odd}=b_{even}$.

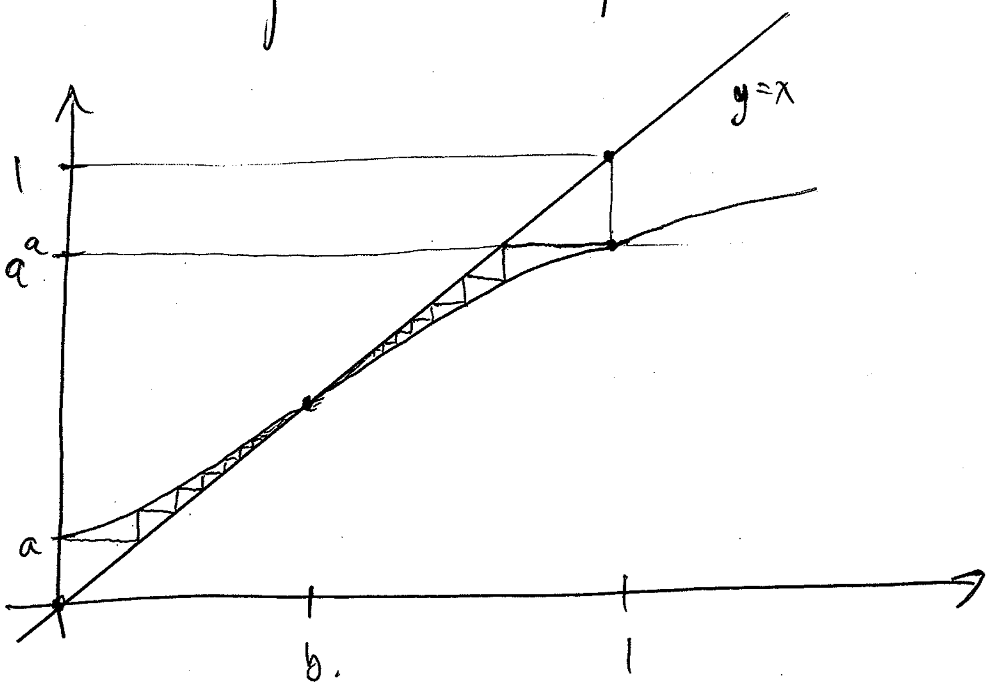
Since $b_{even} = \lim_{n\to\infty} a\uparrow\uparrow(2n) = a^{\lim_{n\to\infty} a\uparrow\uparrow(2n-1)} = a^{b_{odd}}$, we see $a = b_{even}^{\frac{1}{b_{odd}}} = b_{odd}^{\frac{1}{b_{even}}}$, or $\underline{b_{even}^{b_{even}} = b_{odd}^{b_{odd}}}$.

In other words, $\{b_{even}, b_{odd}\}$ forms an Euler pair!

To see the proof continued, consult Anderson.

For graphical evidence of a proof, we can consult <u>Rippon</u>.

In the case that $e^{-e} \le a < 1$, the graphs of $y = x$ and $y = a^{a^x}$ are plotted below:

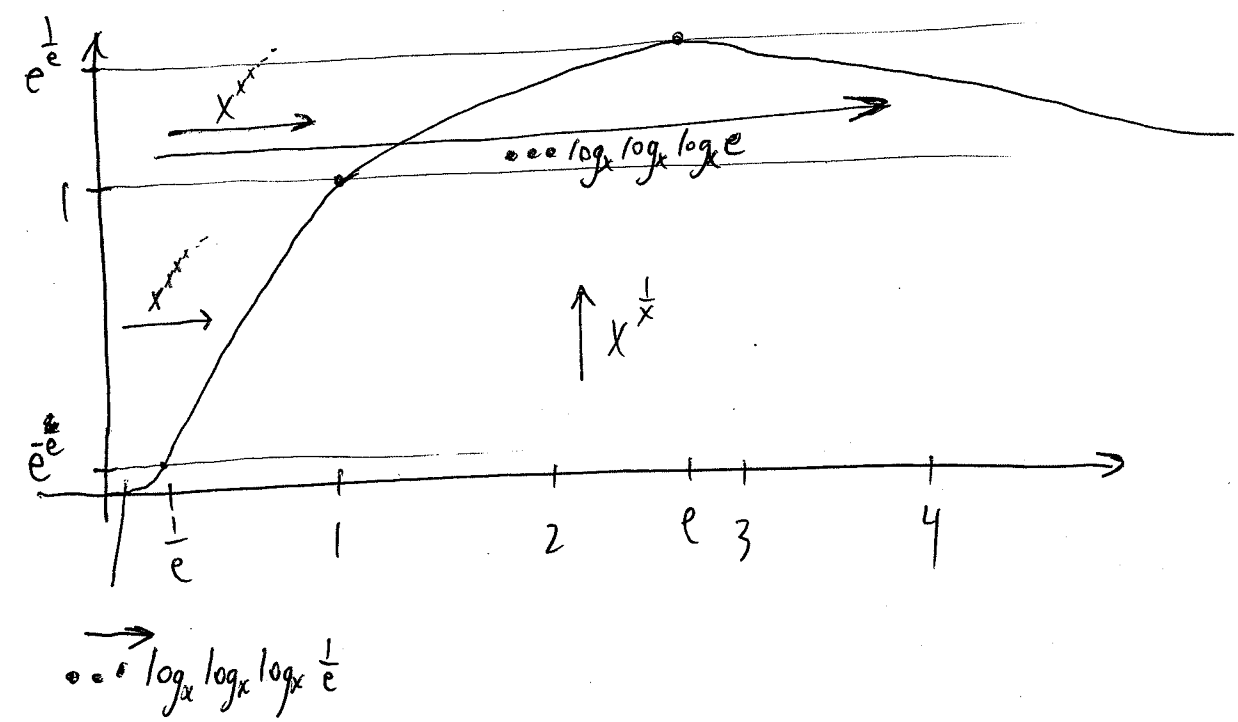


Using elementary calculus, one can show that a^{a^x} has a single point of inflection and its graph crosses $y = x$ exactly once.

In case $0 < a < e^{-e}$, similar graphical analysis leads one to guess that $a, a^a, a^{a^a}, \dots$ does not converge ////

Some loose ends :

Infinite exponentiation $a^{a^{a^{\cdots}}}$ converges for $e^{-e} \leq a \leq e^{\frac{1}{e}}$ and is the inverse of $f(x) = x^{\frac{1}{x}}$ on this interval. Can we describe the inverse of $f(x)$ on $(0, e^{-e})$ or the other branch of the inverse ~~of~~ on $(1, e^{\frac{1}{e}})$?



Incredibly, both questions are answered by the infinite application of $\log_x$ to e and $\frac{1}{e}$ as the graph above indicates!

This is the work of Cho and Park in 2001.

Finally, it is a natural question to consider extending the domain of the repeated exponential $a^{a^{a}}$ to complex numbers. In this case $z^{w} = e^{w \log z}$ depends on the branch of the logarithm chosen, so exponentiation is no longer single valued. Still, Thron in 1957 showed $z^{z^{z^{\cdots}}}$ converges for $|\log z| \le \frac{1}{e}$. Many open problems remain regarding this convergence. See Knoebel for many references and a discussion.

[D. Shell showed in his PhD thesis that $i^{i^{i^{\cdots}}}$ converges!].

References:

- Euler (see page 1)
- ✱ J. Anderson, Iterated Exponentials, Am. Math. Monthly, Vol. 111, No. 8 (2004)
- P. J. Rippon, Infinite exponentials, Math. Gazette, Vol. 67, No. 441 (1983
- R. Arthur Knoebel, Exponentials Reiterated, Am. Math. Monthly, Vol. 88, No 4 (1981).

  Knoebel has an extensive list of references on this topic

✱ : My primary reference.