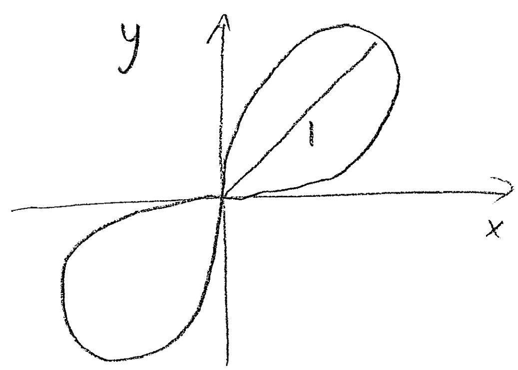
Euler's elliptic integral addition theorem

Count Giulio Carlo de'Toschi di Fagnano (1682–1766) was an Italian amateur mathematician. He published a series of papers from 1714–1718 on the lengths of various curves (a topic introduced by Johann Bernoulli in 1698).

Perhaps most notably, he considered the arc length of the lemniscate

$$(x^2+y^2)^2 = 2xy$$

$$r^2 = \sin(2\theta)$$

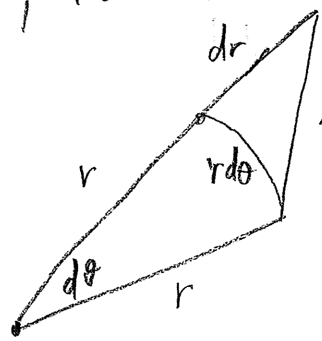


Length of the lemniscate is 5.24411... called the lemniscate constant.

Jacob Bernoulli may have been the first to consider the arc length of the lemniscate as early as 1691. He described the curve as "like a lying eight-like figure, folded in a knot of a bundle, or of a lemniscus, a knot of a French ribbon."

Also, lemniskos is a knot in the form of an eight.

To compute its length, it's best to use the polar form and recall that



$$\sim \sqrt{(dr)^2 + r^2 (d\theta)^2} = \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr.$$

Differentiating $r^2 = \sin(2\theta)$ yields $2r\,dr = 2\cos(2\theta)\,d\theta$, whereby $\dfrac{d\theta}{dr} = \dfrac{r}{\cos(2\theta)} = \dfrac{r}{\sqrt{1-r^4}}$. Therefore, the length of the lemniscate is

$$4 \cdot \int_0^1 \sqrt{1 + r^2 \frac{r^2}{1-r^4}}\, dr = \boxed{4 \int_0^1 \frac{dr}{\sqrt{1-r^4}}} \approx 5.24411\ldots \quad ⊛$$

[The integrand $\frac{1}{\sqrt{1-x^4}}$ does not have an anti-derivative expressible in elem. functions!]

The integral $\displaystyle\int \frac{dx}{\sqrt{1-x^4}}$ is an example of an <u>elliptic integral</u>.

An elliptic integral is an integral of the form

$$\int R(x, \sqrt{f(x)})\, dx$$

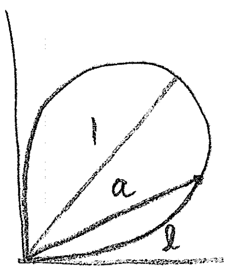
where R is a rational function and $f(x)$ is a polynomial of degree 3 or 4. (When $\deg f > 4$, they are called "hyperelliptic")

Elliptic integrals arise naturally in computing arc lengths (ellipse, lemniscate, ...), calculating the period of a simple pendulum, etc...

⊛ In what follows, we will ignore for the sake of exposition the technicalities of improper integrals & functions and their inverses!

In exploring the integral, Fagnano discovered the following:

$$\ell = \int_0^a \frac{dx}{\sqrt{1-x^4}} \underset{\uparrow}{=} \frac{1}{\sqrt{2}} \int_0^{\frac{\sqrt{2}\,a}{\sqrt{1-a^4}}} \frac{dy}{\sqrt{1+y^4}} \underset{\uparrow}{=} \frac{1}{2} \int_0^{\frac{2a\sqrt{1-a^4}}{1+a^4}} \frac{dz}{\sqrt{1-z^4}}.$$



$$y = \frac{\sqrt{2}\,x}{\sqrt{1-x^4}} \qquad z = \frac{\sqrt{2}\,y}{\sqrt{1+y^4}}$$

$$x = \frac{\sqrt{\sqrt{1+y^4}-1}}{y} \qquad y = \frac{\sqrt{\sqrt{1-z^4}+1}}{z}.$$

Note that given a, $\frac{2a\sqrt{1-a^4}}{1+a^4}$ is constructable with ruler & compass!

So, we have <u>Fagnano's addition theorem</u>. ($\approx$ 1915).

$$\boxed{2\int_0^a \frac{dx}{\sqrt{1-x^4}} = \int_0^{\frac{2a\sqrt{1-a^4}}{1+a^4}} \frac{dz}{\sqrt{1-z^4}}}$$

"doubling the arc length of the lemniscate"

which expresses twice the length of a segment of the lemniscate as another length along the lemniscate.

=

Side note: Fagnano's theorem actually reads:

$$\frac{\sqrt{2}\,x}{\sqrt{1-x^4}} = \frac{\sqrt{1-\sqrt{1-z^4}}}{z} \text{ is a solution to } \frac{2dx}{\sqrt{1-x^4}} = \frac{dz}{\sqrt{1-z^4}}.$$

This implies the result above by solving for $z = \frac{2x\sqrt{1-x^4}}{1+x^4}$,

and then making the substitution: $\displaystyle\int_0^a \frac{2dx}{\sqrt{1-x^4}} = \int_0^{\frac{2a\sqrt{1-a^4}}{1+a^4}} \frac{dz}{\sqrt{1-z^4}}$.

This addition theorem is more familiar to you than you may think! Indeed, we already used in this talk the little brother of this theorem: $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\sin\theta\sqrt{1-(\sin\theta)^2}.$$

To see why, recall that the sine function is just the inverse of the arcsine integral:

$$\theta = \int_0^{\sin\theta} \frac{dx}{\sqrt{1-x^2}}$$

Since $\theta + \theta = 2\theta$,

$$\int_0^{\sin\theta} \frac{dx}{\sqrt{1-x^2}} + \int_0^{\sin\theta} \frac{dy}{\sqrt{1-y^2}} = \int_0^{\sin(2\theta)} \frac{dz}{\sqrt{1-z^2}}$$

or

$$\boxed{2\int_0^{a} \frac{dx}{\sqrt{1-x^2}} = \int_0^{2a\sqrt{1-a^2}} \frac{dz}{\sqrt{1-z^2}}}$$ "double angle formula"

= Two ideas stemming from this connection:

1) Just as we study today the sine function more than the arcsin function, it is extremely fruitful to invert $a \longmapsto \int_0^a \frac{dx}{\sqrt{1-x^4}}$.

We will discuss this shortly.

2) What about the analoge to $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$?

What about a lemniscate analogue to $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$?

In comes Euler. Fagnano sent his works to the Berlin Academy, and on Dec. 23, 1751, Euler received them. Carl Gustav Jacob Jacobi described later this day as the "birthday" of elliptic functions. In considering the lemniscate integral, Euler discovered

Theorem (Euler ~1753)

The general solution to the differential equation $\frac{dx}{\sqrt{1-x^4}} = \frac{dz}{\sqrt{1-z^4}}$ is

$$x^2+z^2+b^2x^2z^2 = b^2 + 2xz\sqrt{1-b^4}, \quad b \in \mathbb{R}.$$

To write this as an addition theorem, solve for z:

$$z = \frac{x\sqrt{1-b^4} + b\sqrt{1-x^4}}{1+b^2x^2}$$

and substitute:

$$\int_0^a \frac{dx}{\sqrt{1-x^4}} = \int_b^{\frac{a\sqrt{1-b^4}+b\sqrt{1-a^4}}{1+a^2b^2}} \frac{dz}{\sqrt{1-z^2}} = \int_0^{\frac{a\sqrt{1-b^4}+b\sqrt{1-a^4}}{1+a^2b^2}} \frac{dz}{\sqrt{1-z^4}} - \int_0^b \frac{dy}{\sqrt{1-y^4}},$$

so

$$\boxed{\int_0^a \frac{dx}{\sqrt{1-x^4}} + \int_0^b \frac{dy}{\sqrt{1-y^4}} = \int_0^{\frac{a\sqrt{1-b^4}+b\sqrt{1-a^4}}{1+a^2b^2}} \frac{dz}{\sqrt{1-z^4}}}$$

When $b=a$, we recover Fagnano's theorem.

Just as we inverted the arcsine integral, we may invert the lemniscate integral: define the "sine of the lemniscate," sl, by

$$\alpha = \int_0^{sl(\alpha)} \frac{dx}{\sqrt{1-x^4}} = sl\left(\int_0^{\alpha} \frac{dx}{\sqrt{1-x^4}}\right).$$

Gauss wrote: $sl = \sin \text{lem}$.

Then, since $\alpha + \beta = (\alpha+\beta)$, we get

$$\int_0^{sl(\alpha)} \frac{dx}{\sqrt{1-x^4}} + \int_0^{sl(\beta)} \frac{dy}{\sqrt{1-y^4}} = \int_0^{sl(\alpha+\beta)} \frac{dz}{\sqrt{1-z^4}},$$

where $\boxed{sl(\alpha+\beta) = \dfrac{sl(\alpha)\sqrt{1-sl(\beta)^4} + sl(\beta)\sqrt{1-sl(\alpha)^4}}{1+sl(\alpha)^2 sl(\beta)^2}.}$

= The function sl is an example of an <u>elliptic function</u>. An elliptic function* is the inverse function to an elliptic integral. The idea of studying the inverse functions to elliptic integrals was due first to Gauss in his personal math diary in 1796 (at the latest). Abel was the first to publish on elliptic functions, and C. Jacobi along with Abel & Gauss developed the theory. It seems that Euler did not have the idea of inverting elliptic integrals.

= * A more modern definition: a meromorphic, doubly-periodic, $\mathbb{C}$-valued function on $\mathbb{C}$.

+ Some more details on Eulers work in [2] —

Euler also considered in 1753 the much more general differential equation $\frac{dx}{\sqrt{P(x)}} = \frac{dz}{\sqrt{P(z)}}$, where $P(x)$ is a 4^{th} degree real polynomial without repeated roots, obtaining a similar addition theorem.

He reworked this theorem around 1765 with the advent of the fractional linear transformation $\tilde{x} = \frac{ax+b}{cx+d}$ to simplify $\frac{dx}{\sqrt{P(x)}}$.

<u>Theorem</u> (Euler ~1765) a,b,c,d may be chosen so that if $\hat{x} = \frac{ax+b}{cx+d}$, then $\frac{dx}{\sqrt{P(x)}} = (ad-bc)\frac{d\tilde{x}}{\sqrt{\tilde{P}(\tilde{x})}}$, where $\tilde{P}(\tilde{x})$ is quadratic in $\tilde{x}^2$ and does not have repeated roots.

Then, $(ad-bc)\frac{d\hat{x}}{\sqrt{\tilde{P}(\hat{x})}} = (ad-bc)\frac{d\tilde{z}}{\sqrt{\tilde{P}(\tilde{z})}}$ if and only if $\frac{dx}{\sqrt{P(x)}} = \frac{dz}{\sqrt{P(z)}}$, so we may assume $P(x)$ is quadratic in x^2.

<u>Theorem (Euler ~1753)</u> Suppose $P(x) = 1 + mx^2 + nx^4$.
The general solution to the differential equation $\frac{dx}{\sqrt{P(x)}} = \frac{dz}{\sqrt{P(z)}}$ is $x^2 + z^2 - nb^2x^2z^2 = b^2 + 2xz\sqrt{P(b)}$, $b \in \mathbb{R}$.

In the same way as before, this yields

$$\boxed{\int_0^a \frac{dx}{\sqrt{P(x)}} + \int_0^b \frac{dy}{\sqrt{P(y)}} = \int_0^{\frac{a\sqrt{P(b)}+b\sqrt{P(a)}}{1-na^2b^2}} \frac{dz}{\sqrt{P(z)}}}$$

for $P(x) = 1 + mx^2 + nx^4$. When $m=0$, $n=-1$, we recover the prev. result.

By inverting the fractional linear transformation taking a general 4th degree $P \in \mathbb{R}[X]$ without repeated roots to one which is quadratic in its argument, we can recover similar addition theorems for the elliptic function resulting from inverting $a \mapsto \int_0^a \frac{dx}{\sqrt{P(x)}}$ for a general P!

There are many more examples of addition theorems in the study of elliptic integrals and functions due to Legendre, Abel, Jacobi, Gauss, and Weierstrass. Each of Legendre's 3 types of elliptic integrals has an addition theorem; for example, Legendre's addition theorem for elliptic integrals of the first kind is:

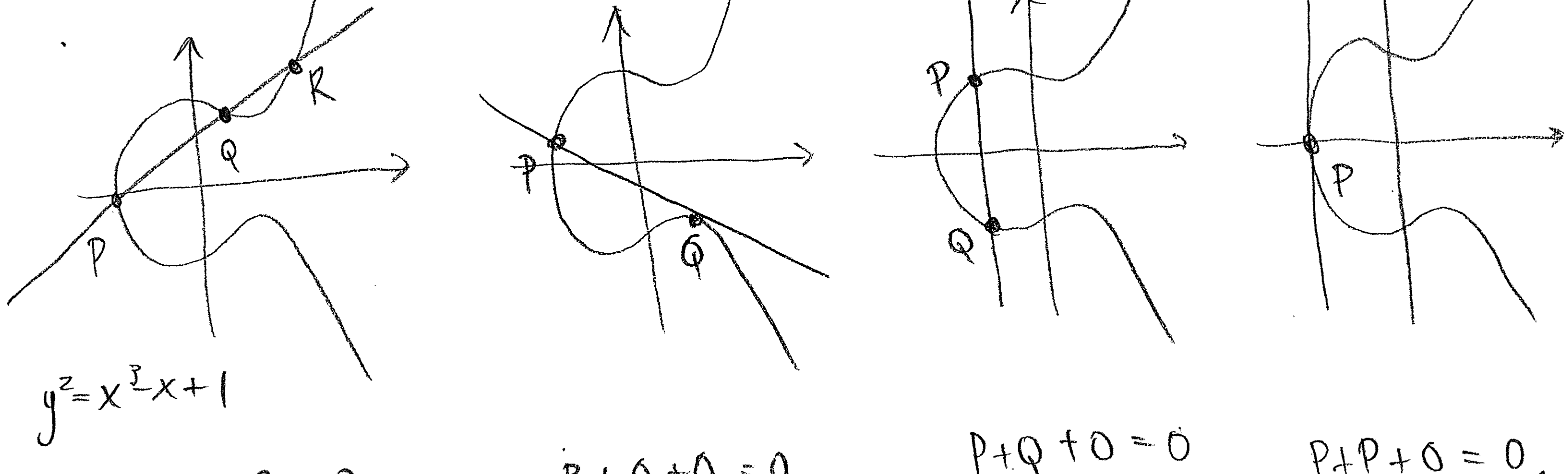
$$\int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2(\sin\phi)^2}} + \int_0^{\psi} \frac{d\psi}{\sqrt{1-k^2(\sin\psi)^2}} = \int_0^{\mu} \frac{d\mu}{\sqrt{1-k^2(\sin\mu)^2}}$$

where $\sin\mu = \dfrac{\sin\phi\cos\psi\sqrt{1-k^2(\sin\psi)^2} + \cos\phi\sin\psi\sqrt{1-k^2(\sin\phi)^2}}{1-k^2(\sin\phi)^2(\sin\psi)^2}$.

- It is interesting to draw analogies between the "classical theory" and this higher order "elliptic theory" (table taken from [4]).

	functions	geometry	arithmetic
classical	arc sin / arc cos sine / cosine	conics (circles, parabolas, ..) $g(x,y)=0$, $\deg g=2$	Pythagorean triples $x^2+y^2=z^2$
elliptic	elliptic integrals elliptic functions	elliptic curves $g(x,y)=0$ $\deg g=3$	Rational points on elliptic curves.

- An elliptic curve* is the locus of points described by $y^2=P(x)$ where $P(x)$ is a polynomial of degree 3 or 4 with no repeated roots. Elliptic curves are actually abelian groups! When $P(x)$ is of the form x^3+ax+b, the addition can be given geometrically:



$y^2=x^3-x+1$

$P+Q+R=0$ $\qquad$ $P+Q+Q=0$ $\qquad$ $P+Q+O=O$ $\qquad$ $P+P+O=O$.

* Elliptic curve: smooth projective algebraic curve of genus 1 w/ specified point.

- It is interesting to connect addition theorems for elliptic functions to the group law for points on elliptic curves.

- As a model, note that the circle $y^2 = 1 - x^2$ is parameterized by the sine function and its derivative cosine: $\theta \mapsto (\sin\theta, \cos\theta)$. Then, addition on the circle is nothing more than addition of the arguments: $(\sin\theta, \cos\theta) + (\sin\alpha, \cos\alpha) = (\sin(\theta+\alpha), \cos(\theta+\alpha))$.
Or, using the addition theorems for sine and cosine, we have
$$(x_1, y_1) + (x_2, y_2) = (x_1 y_2 + x_2 y_1, \ y_1 y_2 - x_1 x_2).$$

- In the same way, elliptic functions parameterize elliptic curves, and the group law for points on an elliptic curve can be described via the addition law for the parameterizing elliptic functions!
Specifically, the elliptic curve $y^2 = P(x)$ is parameterized as $\alpha \mapsto (s(\alpha), s'(\alpha))$ where $s^{-1}(\alpha) = \int_0^\alpha \frac{dx}{\sqrt{P(x)}}$. We don't show this fact here. To see at least that $(s(\alpha), s'(\alpha))$ is on the curve $y^2 = P(x)$, we must check that $(s'(\alpha))^2 = P(s(\alpha))$.

From $s^{-1}(\alpha) = \int_0^\alpha \frac{dx}{\sqrt{P(x)}}$, we see $\frac{1}{s'(s^{-1}(\alpha))} = \frac{1}{\sqrt{P(\alpha)}}$

whereby $s'(\alpha) = \sqrt{P(s(\alpha))}$, and so $(s'(\alpha))^2 = P(s(\alpha))$.

Then, to add two points on the elliptic curve $y^2 = P(x)$,

$$(s(\alpha), s'(\alpha)) + (s(\theta), s'(\theta)) = (s(\alpha+\theta), s'(\alpha+\theta)).$$

If (x_1, y_1), (x_2, y_2) are points on the curve, then the x-coordinate of their sum is x_3 = algebraic in x_1, x_2 given by the addition formula of the corresponding elliptic integral:

$$\int_0^{x_1} \frac{dx}{\sqrt{P(x)}} + \int_0^{x_2} \frac{dx}{\sqrt{P(x)}} = \int_0^{x_3} \frac{dx}{\sqrt{P(x)}}!$$

References

[1] Norman Alling, Real Elliptic curves. North Holland Math Studies 54, 1981

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[2] Euler E251 De integratione aequationis differentialis $\frac{m\,dx}{\sqrt{1-x^4}} = \frac{n\,dy}{\sqrt{1-y^4}}$, presented 1753, published 1761 (20 pages)

E345 Integratio aequationis $\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} = \frac{dy}{\sqrt{A+By+Cy^2+Dy^3+Ey^4}}$, presented in 1765, published 1768 (14 pages)

[3] A. I. Markushevich, Intro to the classical theory of abelian functions

[4] Jan Nekovář, Classical intro to elliptic functions and elliptic curves (www.math.jussieu.fr/~nekovar/co/ln/el).