Euler's Cotangent Series & the Herglotz Trick

Euler derived a beautiful series formula for the cotangent function in his Introductio in Analysin Infinitorum (1748):

$$(*) \qquad \pi \cot(\pi x) = \cdots + \frac{1}{x-2} + \frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \cdots$$

The goal of this talk is to give two separate proofs of this formula: Euler's original, and a proof due to Gustav Herglotz from around the 1930's.

The series in (*) is not absolutely summable. To be clear, here is the definition of the series more explicitly along with an alternate expression which will be useful later on:

$$\cdots + \frac{1}{x-2} + \frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \cdots \quad \text{is defined as:}$$

$$\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + \cdots$$

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{x+n}$$

$$= \frac{1}{x} + \lim_{N\to\infty} \sum_{n=1}^{N} \frac{2x}{x^2-n^2}.$$

Eulers' original proof

Euler's proof has two main steps:

1) Write $\cos y - \cot x \sin y$ as the infinite product

(‡) $$\left(1-\frac{y}{x}\right)\left(1+\frac{y}{\pi-x}\right)\left(1-\frac{y}{\pi+x}\right)\left(1+\frac{y}{2\pi-x}\right)\left(1-\frac{y}{2\pi+x}\right)\cdots$$

2) Using the Taylor series for sine and cosine, and expanding the infinite product, equate the coefficients on y to recover (*)

In Chapter IX of "Analysin" (On Trinomial Factors), Euler initiates the study of finding factors of polynomials (and power series!) which consist of 3 terms (hence, "trinomial"). En route to (‡), he writes in §160: "It would be convenient to find an infinite product expansion for the function $e^{b+x} \pm e^{c-x}$."

To get the flavor of his technique without all the additional variables, let's see how he finds an infinite product expansion for $e^x - e^{-x}$ a few sections earlier.

In §156, Euler writes that

$$e^x - e^{-x} = \left(1+\frac{x}{n}\right)^n - \left(1-\frac{x}{n}\right)^n$$

for n an "infinitely large number." A few sections prior, in §151, Euler tells us that when n is odd,

$$a^n - b^n = (a-b)\prod_{k=1}^{\frac{n-1}{2}} \underbrace{\left(a^2 - 2ab\cos\left(\frac{2\pi k}{n}\right) + b^2\right)}_{\text{trinomial factor}}$$

and, therefore,

$$e^x - e^{-x} = \frac{2x}{n}\prod_{k=1}^{\frac{n-1}{2}} \underbrace{\left(\left(1+\frac{x}{n}\right)^2 - 2\left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}\right)\cos\left(\frac{2\pi k}{n}\right) + \left(1-\frac{x}{n}\right)^2\right)}_{(T)} .$$

["The other terms in the series are neglected since n is infinitely large" §155.]

Since n is an infinitely large number,

$$\cos\left(\frac{2\pi k}{n}\right) \overset{E}{=} 1 - \frac{\left(\frac{2\pi k}{n}\right)^2}{2!} \quad (+ \cdots \text{ is ignored}) .$$

With some algebra, the trinomial factor (T) above becomes

$$\frac{4k^2\pi^2}{n^2}\left(1 + \frac{x^2}{k^2\pi^2} - \cancel{\frac{x^2}{n^2}}\right) \overset{E}{=} \frac{4k^2\pi^2}{n^2}\left(1 + \frac{x^2}{k^2\pi^2}\right),$$

where $\frac{x^2}{n^2}$ may be neglected again since n is infinitely large.

Plugging this back into the product formula,

$$\frac{e^x-e^{-x}}{2} = \frac{x}{n}\prod_{k=1}^{\frac{n-1}{2}} \frac{4k^2\pi^2}{n^2}\left(1+\frac{x^2}{k^2\pi^2}\right) = \underbrace{\frac{1}{n}\left(\frac{4\pi^2}{n^2}\right)^{\frac{n-1}{2}}\left(\left(\frac{n-1}{2}\right)!\right)^2}\cdot x\prod_{k=1}^{\frac{n-1}{2}}\left(1+\frac{x^2}{k^2\pi^2}\right)$$

Euler now writes that this means: $\left(\sim\left(\frac{\pi}{e}\right)^n \text{ as } n\to\infty\text{, though Euler doesn't mention this explicitly}\right)$

$$\frac{e^x-e^{-x}}{2} = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\cdots$$

an infinite product expansion for $\frac{e^x-e^{-x}}{2}$!

[Euler remarks at this point that upon substituting ix for x, we obtain

$$\sin(x) = \frac{e^{ix}-e^{-ix}}{2i} = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

which, because exactly the roots of sine appear on the RHS, "is so obvious, that we might have found the factors from this fact" (§158)]

(5)

In exactly the same way, Euler derives the following in §162:

$$\frac{e^y - e^{2x}e^{-y}}{1-e^{2x}} = \left(1-\frac{y}{x}\right)\left(1-\frac{xy-y^2}{\pi^2+x^2}\right)\left(1-\frac{xy-y^2}{(2\pi)^2+x^2}\right)\cdots$$

Then, through some clever manipulations, the LHS becomes

$$\frac{e^y - e^{2x}e^{-y}}{1+e^{2x}} \cdot \frac{1-e^{-2x}}{1-e^{-2x}} = \frac{e^y + e^{-y} - (e^{y-2x} + e^{-(y-2x)})}{2-(e^{2x}+e^{-2x})}$$

$$= \frac{\frac{e^y+e^{-y}}{2} - \frac{e^{y-2x}+e^{-(y-2x)}}{2}}{1-\frac{e^{2x}+e^{-2x}}{2}}$$

Replacing x with ix and y with iy, this becomes:

$$\frac{\cos y - \cos(y-2x)}{1-\cos(2x)} = \frac{\cos y - \cos y\cos(2x) - \sin y\sin(2x)}{1-\cos(2x)}$$

$$= \cos y - \frac{\sin(2x)}{1-\cos(2x)}\sin y$$

$$= \cos y - \cot(x)\sin y.$$

With this same substitution the product on the RHS is:

$$\left(1-\frac{iy}{ix}\right)\left(1-\frac{ix\cdot iy-(iy)^2}{\pi^2+(ix)^2}\right)\cdots = \left(1-\frac{y}{x}\right)\left(1+\frac{y}{\pi-x}\right)\left(1-\frac{y}{\pi+x}\right)\left(1+\frac{y}{2\pi-x}\right)\left(1-\frac{y}{2\pi+x}\right)\cdots$$

Thus we have a product expansion, the first step:

$$\cos y - \cot x \sin y = \left(1-\frac{y}{x}\right)\left(1+\frac{y}{\pi-x}\right)\left(1-\frac{y}{\pi+x}\right)\left(1+\frac{y}{2\pi-x}\right)\left(1-\frac{y}{2\pi+x}\right)\cdots$$

Step 2 is to expand each side as an infinite series in y and equate the coefficients on the y term.

Using the Taylor series for sine and cosine, the LHS is:

$$\left(1-\frac{y^2}{2!}+\frac{y^4}{4!}-\cdots\right)-\cot x\left(y-\frac{y^3}{3!}+\frac{y^5}{5!}-\cdots\right)$$

$$= 1-(\cot x)y-\frac{y^2}{2!}+\cdots$$

The first 2 terms in the infinite product on the RHS are:

$$1-\left(\frac{1}{x}-\frac{1}{\pi-x}+\frac{1}{\pi+x}-\frac{1}{2\pi-x}+\frac{1}{2\pi+x}-\cdots\right)y+\cdots$$

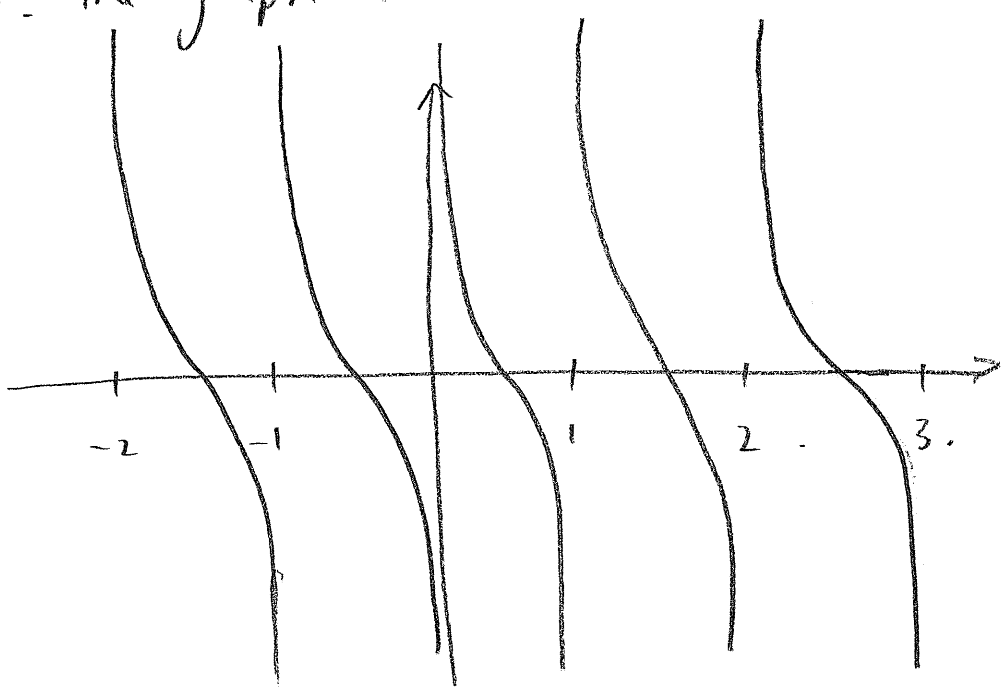
And so,

$$\cot x = \frac{1}{x}+\frac{1}{x-\pi}+\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}-\cdots$$

$$\boxed{\pi\cot(\pi x) = \frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x-2}+\frac{1}{x+2}+\cdots}$$

[Euler did not set out to prove this series for cotangent specifically – it came out naturally in the course of developing many related infinite product / sum formulas. As such, this presentation is a fast-tracked version of Euler's! Euler writes this series in §178, drawing on work from sections §143 – §177.]

P.S. The graph of $f(x) = \pi \cot(\pi x)$ looks like this:



Herglotz' proof

Gustav Herglotz (1881–1953; German-Bohemian Mathematician) discovered an alternate proof of Euler's partial fraction expansion of the cotangent function. He used an ingenious argument, the final trick of which has come to be known as the "Herglotz Trick". Herglotz may never have published his trick; it appeared first on his lecture notes from around 1930 (see Reference 3). The following proof is taken nearly verbatim from "Proofs from the Book" (see Reference 1)

To prove that

$$\pi \cot(\pi x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{x+n}$$

set $f(x) = \pi \cot(\pi x)$ and $g(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{x+n}$.

The strategy is to show that f and g both satisfy a list of properties robust enough to conclude that $f=g$.

First, some basic properties:

(1) $f, g : \mathbb{R} \setminus \mathbb{Z} \longrightarrow \mathbb{R}$ are both continuous.

- This is immediate for $f(x) = \pi \cot(\pi x) = \pi \dfrac{\cos(\pi x)}{\sin(\pi x)}$ by the basic properties of the sine and cosine functions.

- To show continuity of $g(x)$, let $x_0 \in \mathbb{R} \setminus \mathbb{Z}$. Since the uniform limit of a sequence of continuous functions is continuous, it suffices to show that $g_N(x) = \sum_{n=-N}^{N} \frac{1}{x+n}$ converges uniformily to $g(x)$ on some nbhd of x_0.

By combining $\frac{1}{x+n} + \frac{1}{x-n} = \frac{2x}{x^2-n^2}$, we may rewrite $g_N(x) = \frac{1}{x} - 2x \sum_{n=1}^{N} \frac{1}{n^2-x^2}$. Thus, it suffices to prove that $h_N(x) = \sum_{n=1}^{N} \frac{1}{n^2-x^2}$ converges uniformily on some neighborhood $[x_0 - \varepsilon, x_0 + \varepsilon] \subseteq \mathbb{R} \setminus \mathbb{Z}$.

Let N_0 be such that $2N_0 - 1 > (x_0+\varepsilon)^2$. Then for $n \geq N_0$ and $x \in [x_0-\varepsilon, x_0+\varepsilon]$,

$$\frac{1}{n^2-x^2} \leq \frac{1}{n^2-(x_0+\varepsilon)^2} < \frac{1}{n^2-2n+1} = \frac{1}{(n-1)^2}.$$

Therefore, for all $x \in (x_0-\varepsilon, x_0+\varepsilon)$, for all $N > M \geq N_0$,

$$|h_N(x) - h_M(x)| \leq \sum_{n=M+1}^{N} \left|\frac{1}{n^2-x^2}\right| \leq \sum_{n=M+1}^{N} \frac{1}{(n-1)^2}$$

Since $\sum_{n=2}^{\infty} \left(\frac{1}{n-1}\right)^2 < \infty$, we see that $\{h_N|_{[x_0-\varepsilon, x_0+\varepsilon]}\}_{N\in\mathbb{N}}$ is uniformily Cauchy, hence uniformily convergent!

(2) f and g are both periodic of period 1: for all $x \in \mathbb{R}\setminus\mathbb{Z}$, $f(x) = f(x+1)$ and $g(x) = g(x+1)$.

- The cotangent function has period π, hence f has period 1.

To see that g has period 1, note that

$$g_N(x+1) = \sum_{n=-N}^{N} \frac{1}{x+1+n} = \sum_{n=-(N-1)}^{N+1} \frac{1}{x+n} = g_{N-1}(x) + \frac{1}{x+N} + \frac{1}{x+N+1}$$

Then, $g(x+1) = \lim_{N\to\infty} g_N(x+1)$

$$= \lim_{N\to\infty} g_{N-1}(x) + \lim_{N\to\infty}\left(\frac{1}{x+N} + \frac{1}{x+N+1}\right)$$

$$= \lim_{N\to\infty} g_N(x) = g(x).$$

(3) f and g are both odd functions: $f(-x) = -f(x)$,
$g(-x) = -g(x)$.

The cotangent function is odd, hence so is f. Since each $g_N(-x) = \sum_{n=-N}^{N} \frac{1}{-x+n} = -\sum_{n=-N}^{N} \frac{1}{x-n} = -g_N(x)$ is odd, so is g.

We now come to a more interesting property which is the centerpiece of the Herglotz Trick:

(4) f and g both satisfy the same functional equation:

$$f\left(\frac{x}{2}\right)+f\left(\frac{x+1}{2}\right)=2f(x), \quad g\left(\frac{x}{2}\right)+g\left(\frac{x+1}{2}\right)=2g(x).$$

We check:

$$f\left(\frac{x}{2}\right)+f\left(\frac{x+1}{2}\right)=\pi\left(\frac{\cos(\pi\frac{x}{2})}{\sin(\pi\frac{x}{2})}+\frac{\cos(\pi\frac{x}{2}+\frac{\pi}{2})}{\sin(\pi\frac{x}{2}+\frac{\pi}{2})}\right)$$

$\cos(x+\frac{\pi}{2})=\sin(x)$

$\sin(x+\frac{\pi}{2})=-\cos(x)$

$$=\pi\left(\frac{\cos(\pi\frac{x}{2})}{\sin(\pi\frac{x}{2})}-\frac{\sin(\pi\frac{x}{2})}{\cos(\pi\frac{x}{2})}\right)$$

$$=\pi\left(\frac{\cos(\pi\frac{x}{2})^2-\sin(\pi\frac{x}{2})^2}{\frac{1}{2}\cdot 2\sin(\pi\frac{x}{2})\cos(\pi\frac{x}{2})}\right)$$

$\cos(2x)=\cos^2x-\sin^2x$

$\sin(2x)=2\sin x\cos x.$

$$=\pi\left(\frac{\cos(\pi x)}{\frac{1}{2}\sin(\pi x)}\right)=2f(x)$$

For g, note that

$$\frac{1}{\frac{x}{2}+n}+\frac{1}{\frac{x+1}{2}+n}=2\left(\frac{1}{x+2n}+\frac{1}{x+2n+1}\right),$$ whereby

$$g_N\left(\frac{x}{2}\right) + g_N\left(\frac{x+1}{2}\right) = \sum_{n=-N}^{N}\left(\frac{1}{\frac{x}{2}+n} + \frac{1}{\frac{x+1}{2}+n}\right)$$

$$= 2\sum_{n=-N}^{N}\left(\frac{1}{x+2n} + \frac{1}{x+2n+1}\right)$$

$$= 2\sum_{n=-2N}^{2N}\frac{1}{x+n} + \frac{1}{x+2N+1}$$

$$= 2g_{2N}(x) + \frac{1}{x+2N+1}.$$

Taking $N\to\infty$ yields $g\left(\frac{x}{2}\right) + g\left(\frac{x+1}{2}\right) = 2g(x)$.

The goal is to show that $f = g$, so consider the function $h(x) = f(x) - g(x) = \pi\cot(\pi x) - \left(\frac{1}{x} - \sum_{n=1}^{\infty}\frac{2x}{n^2-x^2}\right)$ defined on $\mathbb{R}\setminus\mathbb{Z}$. The function h is (1) continuous, (2) periodic of period 1, and (3) odd. By the linearity of the functional equation, (4) h satisfies $h\left(\frac{x}{2}\right) + h\left(\frac{x+1}{2}\right) = 2h(x)$.

We know one more thing about h:

(5) By setting $h(z)=0$ for all $z\in\mathbb{Z}$, h extends to a continuous function on $\mathbb{R}$.

By periodicity, it suffices to check that $\lim_{x\to 0} h(x) = 0$.

But $\lim_{x\to 0} h(x) = \lim_{x\to 0}\left(\pi\cot(\pi x) - \frac{1}{x}\right) + \underbrace{\lim_{x\to 0}\sum_{n=1}^{\infty}\frac{2x}{n^2-x^2}}_{=0.}$

$$= \lim_{x\to 0}\left(\frac{\pi x\cos(\pi x) - \sin(\pi x)}{x\sin(\pi x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\pi x\left(1-\frac{(\pi x)^2}{2!}+\cdots\right) - \left(\pi x - \frac{(\pi x)^3}{3!}+\cdots\right)}{x\left(\pi x - \frac{(\pi x)^3}{3!}+\cdots\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\frac{(\pi x)^3}{2!} + \frac{(\pi x)^3}{3!}+\cdots}{\pi x^2 - \frac{x(\pi x)^3}{3!}+\cdots}\right)$$ OR, use L'Hospital.

$$= \lim_{x\to 0}\left(\frac{x^3\left(\frac{-\pi^3}{2!}+\frac{\pi^3}{3!}+\cdots\right)}{x^2\left(\pi - \frac{\pi^3 x^2}{3!}+\cdots\right)}\right) = 0.$$

Hence, h is a continuous, 1-periodic, odd function $\mathbb{R}\to\mathbb{R}$, zero on the integers, satisfying $h(\frac{x}{2}) + h(\frac{x+1}{2}) = 2h(x)$.

At last, we finish the proof. To show $f=g$, we must show that $h \equiv 0$. Since $h: \mathbb{R} \to \mathbb{R}$ is continuous, periodic, and zero at 0, it achieves its maximum $h(x_0) = m \geq 0$ at some $x_0 \in [0,1]$. Since h is odd, $m = 0$ if and only if $h \equiv 0$. Now (4) at x_0 gives:

$$h\left(\frac{x_0}{2}\right) + h\left(\frac{x_0+1}{2}\right) = 2h(x_0) = 2m$$

from which it follows that $h\left(\frac{x_0}{2}\right) = h\left(\frac{x_0+1}{2}\right) = m$ since m is the maximum of h. Iterating this, $h\left(\frac{x_0}{2^k}\right) = m$ for all $k \in \mathbb{N}$, and since $\frac{x_0}{2^k} \longrightarrow 0$ as $k \to \infty$ and h is continuous, $h(0) = m$. But $h(0) = 0$, so $m = 0$. ∎

References

1) Aigner, Martin; Ziegler, Günter M. Proofs from the Book. Fourth edition. Springer-Verlag, Berlin, 2010

2) Euler, Leonhard. Introduction to the Analysis of the Infinite. Book I. Translated by John Blanton. Springer-Verlag, New York, 1988.
(originally, Introductio in analysin infinitorum, vol 1, 1748)

3) Remmert, Reinhold. Theory of complex functions. Translated by Robert Burckel. GTM, 122, Readings in Math, Springer-Verlag, New York, 1991